Characterization of random uncertainty of measurements Random variables

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Fizika és Kémia Tanszék

Probability

- If an experiment has a total of *N* possible, random results and a given result *A* occurs *k* times, then the probability of result *A* is:

$$p(A) = \frac{k}{N}$$

(This definition is the classical definition of probability. From now on, we will be satisfied with this!)

Random error

- the random error is theoretically a *random variable*
- a random variable could have:
  - discrete values from the elements of a discrete set, or
  - continuous values from the elements of a continuous interval.
- the sum of the probabilities of all possible values of a random variable is always 1. $\sum_{\forall A} p(A) = 1$

Distribution function

- The function *F(x)*, which gives the probability that the random variable *A* takes a value smaller than *x*, is called *distribution function* of the random variable *A*:

$$F(x) = p(A < x).$$

Density function

- The *probability density function* of the continuous random variable is defined as:

$$\rho(x) = \frac{dF(x)}{dx}$$

- From above definition follows:

$$p(a \le A \le b) = \int_a^b \rho(x)dx$$

$$\int_{\forall A} \rho(x)dx = 1$$

Moments

- Discrete case:

$$M_k = \sum_{\forall i} A_i^k . p(A_i)$$

- Continuous case:

$$M_k = \int_{\forall x} x^k . \rho(x) dx$$

Expected value and standard deviation

- The first moment of a random variable is called *expected value*:

$$m = M_1(x) = \langle x \rangle.$$

- The second moment of deviations of random variable A from it's expected value is called *standard deviation*:

$$D^2 = M_2(x - m) = \left\langle (x - m)^2 \right\rangle.$$

Theorem: $D^2 = M_2(x) - m^2$

Proof:

$$D^2 = M_2(x-m) = \left\langle (x-m)^2 \right\rangle = \left\langle x^2 \right\rangle - 2m\langle x \rangle + m^2 =$$

$$= M_2(x) - 2m^2 + m^2 = M_2(x) - m^2.$$

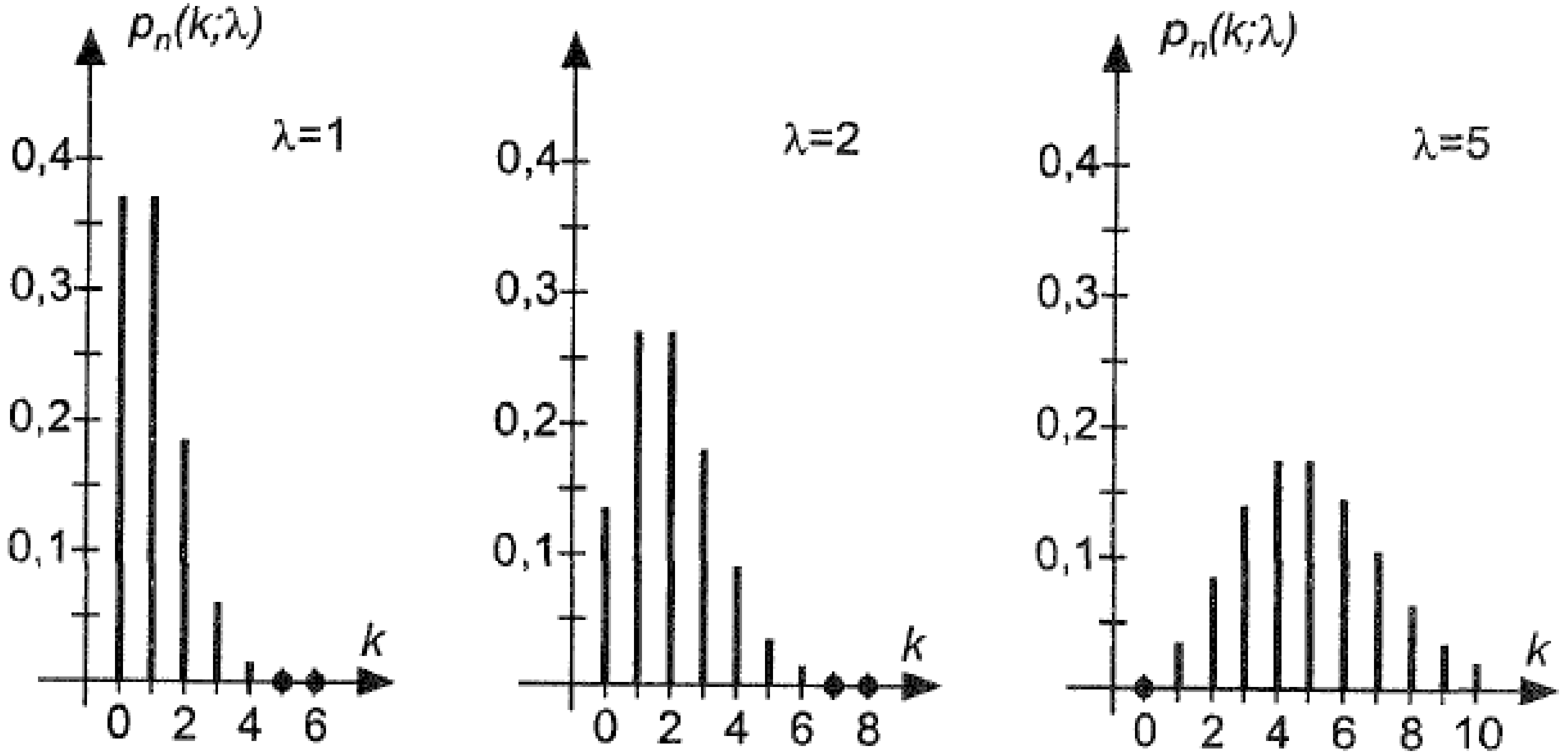
The Poisson distribution

- If *X* is a discrete random variable and take values *0,1,2,…,k* with probability:

$$p(X = k) = \frac{\lambda^k . e^{-\lambda}}{k!} \qquad \lambda > 0$$

than *X* has the *Poisson distribution*

- Examples: the number of raindrops falling on a given area, the average length of the waiting list during a given time period, the number of radioactive decays during a given time inter-val, etc.



- From Calculus we now the Taylor serie for exponential funtion:

$$e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

Characterization of Poisson distribution

- Does it describe probability?

$$\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda}.e^{\lambda} = 1$$

- What is it's expected value?

$$m = \sum_{k=0}^{\infty} k \frac{\lambda^k .e^{-\lambda}}{k!} = 0 + \lambda . \sum_{k=1}^{\infty} \frac{\lambda^{(k-1)} .e^{-\lambda}}{(k-1)!}$$

$$= \lambda .e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} = \lambda .e^{-\lambda} .e^{\lambda} = \lambda$$

Characterization of Poisson distribution

- What is it's standard deviation?:

$$D^2 = \left\langle k^2 \right\rangle - m^2$$

$$\left\langle k^2 \right\rangle = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k . e^{-\lambda}}{k!} = \lambda . \sum_{k=1}^{\infty} k \frac{\lambda^{(k-1)} . e^{-\lambda}}{(k-1)!} = \lambda . \sum_{r=0}^{\infty} (r+1) \frac{\lambda^r . e^{-\lambda}}{r!}$$

$$= \lambda . \left\{ \sum_{r=0}^{\infty} r \frac{\lambda^r . e^{-\lambda}}{r!} + \sum_{r=0}^{\infty} \frac{\lambda^r . e^{-\lambda}}{r!} \right\} = \lambda . \left\{ m + e^{-\lambda} . e^{\lambda} \right\} = \lambda . (\lambda + 1) = \lambda^2 + \lambda$$

$$D^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

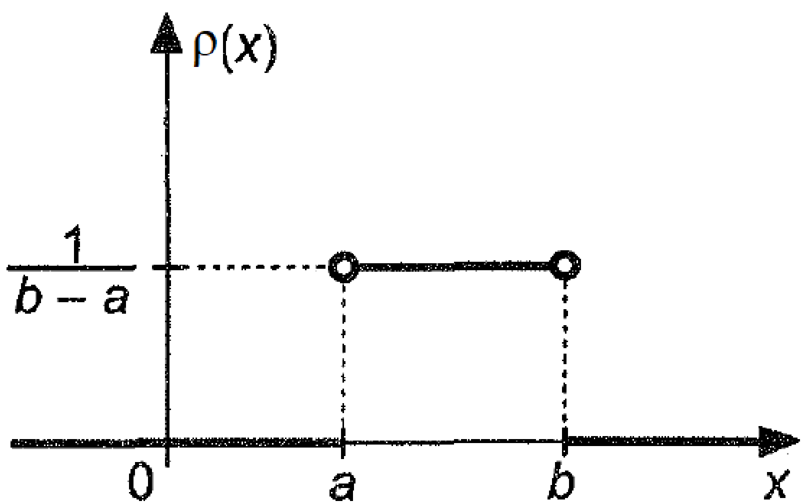
$$m = \lambda$$

$$D^2 = \lambda$$

Uniform distribution

- If X continuous random variable's probability density function is:

$$\rho(x) = \frac{1}{b-a}, \qquad a < x < b$$
$$= 0 \qquad otherwise$$



than random variable is *uniformly distributed*.

- Example: measure-ments with scales

Characterization of uniform distribution

- Does it describe probability?

$$\int_{\forall x} \rho(x)dx = \int_a^b \frac{1}{b-a}dx = \frac{b-a}{b-a} = 1$$

- What is it's expected value?

$$\mu = \int_a^b x.\frac{1}{b-a}dx = \frac{1}{b-a}\int_a^b xdx = \frac{1}{b-a}.\frac{b^2-a^2}{2} = \frac{a+b}{2}$$

Characterization of uniform distribution

- What is it’s standard deviation?:

$$\sigma^2 = \left\langle x^2 \right\rangle - \mu^2$$

$$\left\langle x^2 \right\rangle = \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x^2 dx = \frac{b^3 - a^3}{3(b-a)} = \frac{a^2 + ab + b^2}{3}$$

$$\sigma^2 = \frac{a^2 + ab + b^2}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)}{12}$$

$$\mu = \frac{(a+b)}{2}$$

$$\sigma^2 = \frac{(b-a)^2}{12}$$

Normal distribution or Gaussian distribution

- If probability density function of continuos random variable *X* is (μ_0>0 és σ_0>0)

$$\rho(x) = \frac{1}{\sqrt{2\pi}\sigma_0} . e^{\frac{-(x-\mu_0)^2}{2\sigma_0^2}},$$

than *X* is called *normally distributed* or *Gaussian*.

E.g.: Several independent, small, random deviations make the measured value Gaussian!

Characterization of Normal distribution

- Does it discribe probability?

$$\int_{-\infty}^{\infty} \rho(x)dx = \frac{1}{\sqrt{2\pi}\sigma_0} \int_{-\infty}^{\infty} e^{\frac{-(x-\mu_0)^2}{2\sigma_0^2}} dx$$

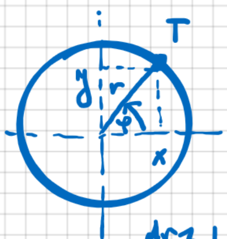
$$t = \frac{x-\mu_0}{\sigma_0}, \frac{dt}{dx} = \frac{1}{\sigma_0}$$

$$\int_{-\infty}^{\infty} \rho(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-t^2}{2}} dt = 1$$

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\, dt = A$$

① $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cdot e^{-\frac{y^2}{2}}\, dx . dy = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx \cdot \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy = A . A = A^2$$

② $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\, dx\, dy = \left\{ \begin{array}{l|l} x = r.\cos\varphi & r^2 = x^2+y^2 \\ y = r.\sin\varphi & dx.dy = r\,dr\,d\varphi \end{array} \right\} = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}}\, r.\, dr\, d\varphi = \left\{ \begin{array}{l} -\frac{r^2}{2} = u \\ \frac{du}{dr} = -r \end{array} \right\} =$$



$r \in [0 ; \infty[$

$\varphi \in [0 ; 2\pi]$

$x = r.\cos\varphi$

$y = r.\sin\varphi$

$\Rightarrow x^2 + y^2 = r^2$

$$= -2\pi \int_0^{-\infty} e^u\, du = -2\pi \left[e^u \right]_0^{-\infty} =$$

$$= -2\pi (0 - 1) = 2\pi = A^2$$

$$A = \sqrt{2\pi}$$

$$\boxed{\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\, dt = \sqrt{2\pi}}$$

Characterization of Normal distribution

- What is it's expected value?

$$\mu = \frac{1}{\sqrt{2\pi}\sigma_0} \int_{-\infty}^{\infty} x.e^{\frac{-(x-\mu_0)^2}{2\sigma_0^2}} dx$$

$$t = \frac{x-\mu_0}{\sigma_0}, x = t.\sigma_0 + \mu_0, \frac{dt}{dx} = \frac{1}{\sigma_0}$$

$$\mu = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (t.\sigma_0 + \mu_0).e^{\frac{-t^2}{2}} dt = \frac{\sigma_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t.e^{\frac{-t^2}{2}} dt + \mu_0 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-t^2}{2}} dt$$

$$\int_{-\infty}^{\infty} t.e^{\frac{-t^2}{2}} dt = 0$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-t^2}{2}} dt = 1$$

$$\mu = \mu_0$$

$$\sigma^2 = M_2(x) - \mu^2 \qquad \rho(x) = \frac{1}{\sqrt{2\pi}\,\sigma_0} \cdot e^{-\frac{(x-\mu_0)^2}{2\sigma_0^2}}$$

$$\boxed{M_2(x)} = \int_{-\infty}^{\infty} x^2 \cdot \rho(x)\, dx =$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_0} \int_{-\infty}^{\infty} x^2 \cdot e^{-\frac{(x-\mu_0)^2}{2\sigma_0^2}}\, dx = \left\{ t = \frac{x-\mu_0}{\sigma_0} \Rightarrow x = \mu_0 + \sigma_0 \cdot t \quad \frac{dx}{dt} = \sigma_0 \right\} =$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_0} \int_{-\infty}^{\infty} (\mu_0 + \sigma_0 t)^2 \cdot e^{-\frac{t^2}{2}} \cdot \sigma_0\, dt = \frac{1}{\sqrt{2\pi}} \left\{ \mu_0^2 \underbrace{\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\, dt}_{=\sqrt{2\pi}} + 2\sigma_0\mu_0 \underbrace{\int_{-\infty}^{\infty} t \cdot e^{-\frac{t^2}{2}}\, dt}_{0 \text{ (integrandus páratlan)}} + \right.$$

$$\left. + \sigma_0^2 \int_{-\infty}^{\infty} t^2 \cdot e^{-\frac{t^2}{2}}\, dt \right\} = \boxed{\mu_0^2 + \frac{\sigma_0^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^2 \cdot e^{-\frac{t^2}{2}}\, dt}$$

$$\int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2}}\, dt = B$$

$$\boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2+y^2) \cdot e^{-\frac{x^2+y^2}{2}}\, dx\, dy} = \left\{ \begin{matrix} x = r\cos\varphi & ; \ x^2+y^2 = r^2 \\ y = r\sin\varphi & ; \ dx\, dy \to r\, dr\, d\varphi \end{matrix} \right\} =$$

$$= \int_0^{2\pi}\int_0^{\infty} r^2 \cdot e^{-\frac{r^2}{2}} \cdot r\, dr\, d\varphi = 2\pi \int_0^{\infty} r^3 \cdot e^{-\frac{r^2}{2}}\, dr = \left\{ \begin{matrix} -\frac{r^2}{2} = u \\ \frac{du}{dr} = u' = -r \end{matrix} \right\} =$$

$$= 2\pi \int_0^{-\infty} 2u \cdot e^{u}\, du = 4\pi \int_0^{-\infty} u \cdot e^{u}\, du = \left\{ \begin{matrix} f = u & f' = 1 \\ g' = e^{u} & g = e^{u} \end{matrix} \right\} =$$

$$= 4\pi \left\{ \underbrace{\left[u \cdot e^{u}\right]_0^{-\infty}}_{=0} - \int_0^{-\infty} e^{u}\, du \right\} = -4\pi \left[e^{u}\right]_0^{-\infty} = \boxed{4\pi}$$

$$\boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2+y^2) \cdot e^{-\frac{x^2+y^2}{2}}\, dx\, dy =}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2+y^2)\, e^{-\frac{x^2}{2}} \cdot e^{-\frac{y^2}{2}}\, dx\, dy = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \underbrace{\int_{-\infty}^{\infty} x^2 \cdot e^{-\frac{x^2}{2}}\, dx}_{=B}\, dy +$$

$$+ \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \underbrace{\int_{-\infty}^{\infty} y^2 \cdot e^{-\frac{y^2}{2}}\, dy}_{=B}\, dx = B \underbrace{\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy}_{=\sqrt{2\pi}} + B \underbrace{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx}_{=\sqrt{2\pi}} = \boxed{2B \cdot \sqrt{2\pi}}$$

$$2B\sqrt{2\pi} = 4\pi$$

$$B\sqrt{2\pi} = (\sqrt{2\pi})^2$$

$$B = \sqrt{2\pi}$$

$$\boxed{B = \int_{-\infty}^{\infty} t^2 \cdot e^{-\frac{t^2}{2}}\, dt = \sqrt{2\pi}}$$

$$\boxed{\sigma^2} = M_2(x) - \mu^2 = \mu_0^2 + \frac{\sigma_0^2}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} t^2 \cdot e^{-\frac{t^2}{2}}\, dt}_{=B=\sqrt{2\pi}} - \mu_0^2 = \boxed{\sigma_0^2}$$

$$(\mu = \mu_0)$$

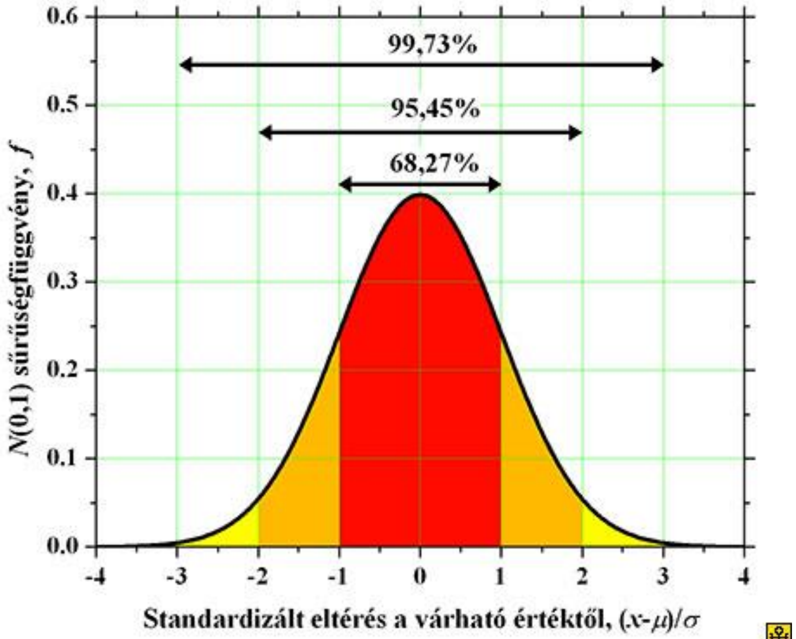
Characterization of Normal distribution

- What is it's standard deviation?

$$\sigma^2 = \left\langle x^2 \right\rangle - \mu^2$$

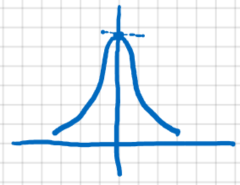
$$\left\langle x^2 \right\rangle = \frac{1}{\sqrt{2\pi}\sigma_0} \int_{-\infty}^{\infty} x^2 . e^{\frac{-(x-\mu_0)^2}{2\sigma_0^2}} dx = \sigma_0^2 + \mu_0^2$$

$$\sigma^2 = \sigma_0^2$$



$$\mu = \mu_0$$
$$\sigma^2 = \sigma_0^2$$

$$\rho(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma_0} \cdot e^{-\frac{(x-\mu_0)^2}{2\sigma_0^2}}$$



① $\rho(x=\mu_0) = \frac{1}{\sqrt{2\pi}\,\sigma_0}$ $\qquad \sigma_0 \nearrow \to \rho(x=\mu_0) \searrow$

② $\rho(x = \mu_0 \pm \sigma_0) \Rightarrow \rho(x) = \frac{\rho(x=\mu_0)}{2}$

$$\frac{1}{\sqrt{2\pi}\,\sigma_0}\, e^{-\frac{(\mu_0 \pm \sigma_0 - \mu_0)^2}{2\sigma_0^2}} = \frac{1}{\sqrt{2\pi}\,\sigma_0}\, e^{-\frac{1}{2}} = \frac{1}{\sqrt{2\pi}\sqrt{e}\cdot\sigma_0} \;\Big|\; \frac{1}{\sqrt{2\pi}\,\sigma_0\cdot 2}$$

✗

③ $\rho(x = \mu_0 \pm \sigma_0) \Rightarrow \rho(x) = \frac{\rho(x=\mu_0)}{e}$

✗ $\;\Big|\; \frac{1}{\sqrt{2\pi}\,\sigma_0\cdot e}$

$x = \mu_0 \pm \sigma_0$ pontok a Gauss-sűrűség-fgv. inflexiós pontjai

$$\rho'(x) = \left[\frac{1}{\sqrt{2\pi}\,\sigma_0}\cdot e^{-\frac{(x-\mu_0)^2}{2\sigma_0^2}}\right]' = \frac{1}{\sqrt{2\pi}\,\sigma_0}\, e^{-\frac{(x-\mu_0)^2}{2\sigma_0^2}} \cdot \left(-\frac{2(x-\mu_0)}{2\sigma_0^2}\right)\cdot 1 =$$

$$= -\frac{(x-\mu_0)}{\sqrt{2\pi}\,\sigma_0^3}\, e^{-\frac{(x-\mu_0)^2}{2\sigma_0^2}}$$

$$\rho''(x) = -\frac{1}{\sqrt{2\pi}\,\sigma_0^3}\left[(x-\mu_0)\cdot e^{-\frac{(x-\mu_0)^2}{2\sigma_0^2}}\right]' = -\frac{1}{\sqrt{2\pi}\,\sigma_0^3}\left[e^{-\frac{(x-\mu_0)^2}{2\sigma_0^2}} + (x-\mu_0)\cdot e^{-\frac{(x-\mu_0)^2}{2\sigma_0^2}}\cdot\left(-\frac{2(x-\mu_0)}{2\sigma_0^2}\cdot 1\right)\right]$$

$$= -\frac{1}{\sqrt{2\pi}\,\sigma_0^3}\, e^{-\frac{(x-\mu_0)^2}{2\sigma_0^2}}\left[1 - (x-\mu_0)\frac{(x-\mu_0)}{\sigma_0^2}\right] = 0 \Leftrightarrow 1 - \frac{(x-\mu_0)^2}{\sigma_0^2} = 0$$

$$\sigma_0^2 - (x-\mu_0)^2 = 0 \Rightarrow (x-\mu_0)^2 = \sigma_0^2 \Rightarrow |x-\mu_0| = \sigma_0 \Rightarrow x = \mu_0 \pm \sigma_0$$

Central limit theorem

- The sum of *N* independent variables (μ_0 ,σ_0) with the same probability distribution is a random variable with a *normal distribution* in the limiting case $N\rightarrow\infty$, the expected value of which is $\mu=N. \mu_0$, and its standard deviation $\sigma^2=N. \sigma_0^2$.