# Vector analysis in nutshell

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## Operators

- mathematical operations with functions

For example:

- the operation of multiplying by a coordinate : $x.$ $(x.f(x) = g(x))$
  - If $f(x) = x^2 \Rightarrow x.f(x) = g(x) = x.x^2 = x^3$
  - If $f(x) = \sin x \Rightarrow x.f(x) = g(x) = x.\sin x$
- coordinate derivative operation: $x'.$
  - If $f(x) = x^2 \Rightarrow (x^2)' = g(x) = 2x$
  - If $f(x) = \sin x \Rightarrow (\sin x)' = g(x) = \cos x$

Vector functions $f(\vec{r}) = f(x;y;z)$ – scalar ; $\vec{V}(\vec{r})$ – vector

Differential operator Nabla: $\vec{\nabla} \equiv \left(\frac{\partial}{\partial x};\frac{\partial}{\partial y};\frac{\partial}{\partial z}\right)$

- gradient $\vec{\nabla} f(x;y;z) \equiv \left(\frac{\partial f}{\partial x};\frac{\partial f}{\partial y};\frac{\partial f}{\partial z}\right)$ – vector
- divergence $\vec{\nabla}.\vec{V}(x;y;z) \equiv \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$ – scalar
- curl $\vec{\nabla} \times \vec{V}$

$$\equiv \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \equiv \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}; -\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right); \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

– vector

Example:

$$f = x^2 + 2y + \frac{3}{z}$$ -

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = 2x\cdot\vec{i} + 2\vec{j} - 3\cdot\frac{1}{z^2}\cdot\vec{k}$$ - scalar function

Example:

$$f_x = x^2 + 2y + 3\sin z$$

$$f_y = 3x - \frac{4}{y} + 5z^2$$

$$f_z = 4\cos x + 3e^y - 4z$$

vector - vector function

$$\vec{\nabla}\cdot\vec{f} = div\,\vec{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} = 2x + \frac{4}{y^2} - 4$$ - scalar function

$$\vec{\nabla}\times\vec{f} = \text{curl}\,\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = \vec{i}\begin{vmatrix} \partial_y & \partial_z \\ f_y & f_z \end{vmatrix} - \vec{j}\begin{vmatrix} \partial_x & \partial_z \\ f_x & f_z \end{vmatrix} + \vec{k}\begin{vmatrix} \partial_x & \partial_y \\ f_x & f_y \end{vmatrix} = \vec{i}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) - \vec{j}\left(\frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z}\right.$$

$$+\vec{k}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) =$$

$$= \vec{i}(3e^y - 10z) - \vec{j}(-4\sin x - 3\cos z) + \vec{k}(3 - 2) = (3e^y - 10z)\vec{i} + \vec{j}(4\sin x + 3\cos z) + \vec{k}$$

vector - vector function

- Gauss - law

$$\oint_A \vec{v}\cdot d\vec{A} = \int_V \operatorname{div}\vec{v}\, dV$$


- Green - law

$$\oint_S \vec{v}\cdot d\vec{s} = \int_A \operatorname{rot}\vec{v}\cdot d\vec{A}$$

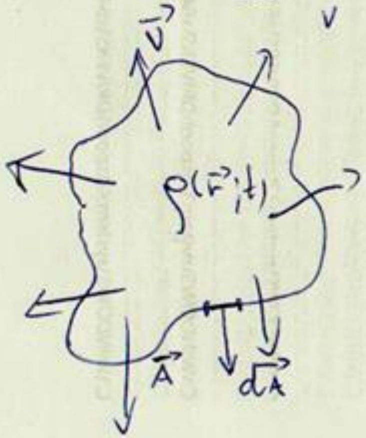

- Conservation of mass:

- $\rho(\vec{r},t)$ - mass density

$m(t) = \int_V \rho(\vec{r},t)\, dV$ - total mass inside of volume V in time t

$$-\frac{dm}{dt} = -\frac{d}{dt}\int_V \rho(\vec{r},t)\, dV = -\int_V \frac{\partial \rho(\vec{r},t)}{\partial t}\, dV = \int_A \rho(\vec{r},t)\cdot \vec{v}(\vec{r},t)\cdot d\vec{A} \overset{G.-O.}{=} \int_V \operatorname{div}(\underbrace{\rho\cdot\vec{v}}_{\vec{j}(\vec{r},t)})\, dV$$

mass flow density




$$\Downarrow$$

$$\int_V \frac{\partial \rho}{\partial t}\, dV + \int_V \operatorname{div}\vec{j}\, dV = \emptyset \quad \forall V$$

$$\int_V \left(\frac{\partial \rho}{\partial t} + \operatorname{div}\vec{j}\right) dV = \emptyset \quad \forall V$$

$$\Downarrow$$

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}\vec{j} = \emptyset}$$ - equation of continuity

Theorem:

$$\operatorname{div}(\operatorname{grad} f) = \Delta f$$

Proof:

$$\vec{u} = \operatorname{grad} f = \left(\frac{\partial f}{\partial x} ; \frac{\partial f}{\partial y} ; \frac{\partial f}{\partial z}\right) = (u_x ; u_y ; u_z)$$

$$\operatorname{div} \vec{u} = \vec{\nabla} \cdot \vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) =$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f$$

The End