Basics of QM I.

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SZÉCHENYI ISTVÁN EGYETEM

Fizika és Kémia Tanszék

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$\hat{H}$ - *Hamilton–operator* of the system
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Let's see how the Hamilton-operator of a mass point is defined!

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$$\boldsymbol{H}(\underline{\boldsymbol{p}}, \underline{\boldsymbol{r}}) = \frac{\underline{\boldsymbol{p}}.\underline{\boldsymbol{p}}}{2\boldsymbol{m}} + \boldsymbol{V}(\underline{\boldsymbol{r}}) = \frac{1}{2\boldsymbol{m}}(\boldsymbol{p}_x^2 + \boldsymbol{p}_y^2 + \boldsymbol{p}_z^2) + \boldsymbol{V}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$$

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$$H(\underline{p}, \underline{r}) = \frac{\underline{p}.\underline{p}}{2m} + V(\underline{r}) = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z)$$

Let's move on to the Hamilton operator:

$$\hat{H} = \frac{1}{2m}(\hat{p}_x\hat{p}_x + \hat{p}_y\hat{p}_y + \hat{p}_z\hat{p}_z) + V(x, y, z).$$

$$\hat{H} = i^2\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) + V(x, y, z).$$

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta + V(x, y, z).$$

Heisenberg's uncertainty relation

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Operators whose commutator is 0 are called compatible operators!

Stationary solutions

The Schrödinger equation of a mass point moving in the time-independent potential field:

$$-\frac{\hbar^2}{2m}\Delta\Psi(x,y,z,t) + V(x,y,z).\Psi(x,y,z,t) = i\hbar\frac{\partial\Psi(x,y,z,t)}{\partial t}$$

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Let's search for the solution in the next form: $e^{-i\frac{E}{\hbar}t}\psi(x,y,z)$. After substitution and simplification:

$$\frac{\hbar^2}{2m}(\Delta - \frac{2m}{\hbar^2}V(x,y,z))e^{-i\frac{E}{\hbar}t}\psi(x,y,z) = -e^{-i\frac{E}{\hbar}t}E\psi(x,y,z)$$

$$\Delta\psi(x,y,z) = -k^2\psi(x,y,z)$$

$$k^2 = \frac{2m(E-V(x,y,z))}{\hbar^2}$$

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$$\frac{\hbar^2}{2m}\left(\Delta-\frac{2m}{\hbar^2}V(x,y,z)\right)e^{-i\frac{E}{\hbar}t}\psi(x,y,z)=-e^{-i\frac{E}{\hbar}t}E\psi(x,y,z)$$

$$\Delta\psi(x,y,z)=-k^2\psi(x,y,z)$$

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Electron on the string

An electron in one dimension if its motion is limited to a section of length a.

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Therefore:

$$E_n = \frac{\hbar^2\pi^2}{2ma^2} n^2$$

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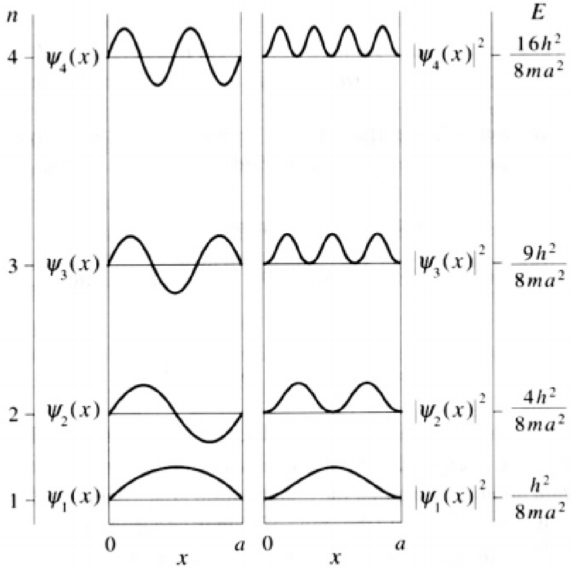
It is generally true that if the movement of a micro-object is limited in space, its energy can only have discrete values.
Note that the smallest energy of the system is not zero, but some positive value. Zero point energy.



n
E
4
$\psi_4(x)$
$|\psi_4(x)|^2$
$\frac{16h^2}{8ma^2}$
3
$\psi_3(x)$
$|\psi_3(x)|^2$
$\frac{9h^2}{8ma^2}$
2
$\psi_2(x)$
$|\psi_2(x)|^2$
$\frac{4h^2}{8ma^2}$
1
$\psi_1(x)$
$|\psi_1(x)|^2$
$\frac{h^2}{8ma^2}$
0
a
x
0
a
x