Vektoranalízis dióhéjban

Dr. Berta Miklós

Operátorok

- matematikai műveletek függvényeken

Pl.: - koordinátával való szorzás művelete: $x.$ $\quad (x.f(x) = g(x))$

- ha $f(x) = x^2 \Rightarrow x.f(x) = g(x) = x.x^2 = x^3$
- ha $f(x) = \sin x \Rightarrow x.f(x) = g(x) = x.\sin x$

- koordináta szerinti deriválás művelete: $x'.$

- ha $f(x) = x^2 \Rightarrow (x^2)' = g(x) = 2x$
- ha $f(x) = \sin x \Rightarrow (\sin x)' = g(x) = \cos x$

Vektorfüggvények: $f(\vec{r}) = f(x;y;z)$ - skalár; $\vec{v}(\vec{r})$ - vektor

Nabla - operátor: $\vec{\nabla} \equiv \left(\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z}\right)$

Gradiens: $\vec{\nabla} f(x;y;z) \equiv \left(\frac{\partial f}{\partial x}; \frac{\partial f}{\partial y}; \frac{\partial f}{\partial z}\right)$ - vektor

Divergencia: $\vec{\nabla}.\vec{v}(x;y;z) \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$ - skalár

Rotáció: $\vec{\nabla} \times \vec{v}$

$$\equiv \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \equiv \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}; -\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right); \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

- vektor

Példa:

$$f = x^2 + 2y + \frac{3}{z} \qquad \text{- skalár függvény}$$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = 2x \cdot \vec{i} + 2\vec{j} - 3 \cdot \frac{1}{z^2} \cdot \vec{k} \qquad \text{- vektor - vektor függvény}$$

Példa:

$$f_x = x^2 + 2y + 3\sin z$$

$$f_y = 3x - \frac{4}{y} + 5z^2 \qquad \text{vektor - vektor függvény}$$

$$f_z = 4\cos x + 3e^y - 4z$$

$$\vec{\nabla} \cdot \vec{f} = \operatorname{div} \vec{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} = 2x + \frac{4}{y^2} - 4 \qquad \text{- skalár függvény}$$

$$\vec{\nabla} \times \vec{f} = \operatorname{rot} \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = \vec{i}\begin{vmatrix} \partial_y & \partial_z \\ f_y & f_z \end{vmatrix} - \vec{j}\begin{vmatrix} \partial_x & \partial_z \\ f_x & f_z \end{vmatrix} + \vec{k}\begin{vmatrix} \partial_x & \partial_y \\ f_x & f_y \end{vmatrix} = \vec{i}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) - \vec{j}\left(\frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z}\right)$$

$$+ \vec{k}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) =$$

$$= \vec{i}\,(3e^y - 10z) - \vec{j}\,(-4\sin x - 3\cos z) + \vec{k}\,(3 - 2) = (3e^y - 10z)\vec{i} + \vec{j}\,(4\sin x + 3\cos z) + \vec{k}$$

vektor - vektor függvény

- Gauss-Osztrogradszkij tétel

$$\oint_A \vec{v}\cdot d\vec{A} = \int_V \mathrm{div}\,\vec{v}\,dV$$

- Stokes-tétel

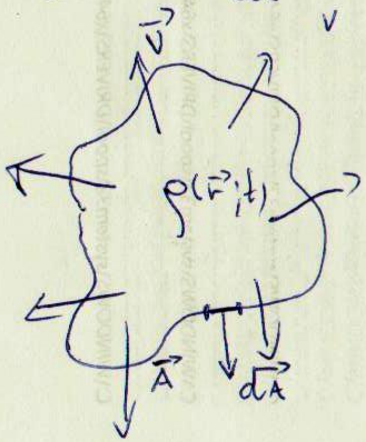
$$\oint_S \vec{v}\cdot d\vec{s} = \int_A \mathrm{rot}\,\vec{v}\cdot d\vec{A}$$

- tömegmegmaradás tétele

- $\rho(\vec{r},t)$ - tömegsűrűség $m(t)=\int_V \rho(\vec{r},t)\,dV$ - össztömeg a V térfogaton belül a t időpillanatban

$$-\frac{dm}{dt} = -\frac{d}{dt}\int_V \rho(\vec{r},t)\,dV = -\int_V \frac{\partial\rho(\vec{r},t)}{\partial t}\,dV = \int_A \rho(\vec{r},t)\cdot\vec{v}(\vec{r},t)\cdot d\vec{A} \overset{G\text{-}O}{=} \int_V \mathrm{div}\,(\underbrace{\rho\cdot\vec{v}}_{\vec{j}(\vec{r},t)})\,dV$$

$\vec{j}(\vec{r},t)$ - tömegáramsűrűség



$$\Downarrow$$

$$\int_V \frac{\partial\rho}{\partial t}\,dV + \int_V \mathrm{div}\,\vec{j}\,dV = 0 \quad \forall V\text{-re}$$

$$\int_V \left(\frac{\partial\rho}{\partial t} + \mathrm{div}\,\vec{j}\right) dV = 0 \quad \forall V\text{-re}$$

$$\Downarrow$$

$$\boxed{\frac{\partial\rho}{\partial t} + \mathrm{div}\,\vec{j} = 0}$$ - kontinuitási egyenlet

Tétel

$$\operatorname{div}(\operatorname{grad} f) = \Delta f$$

Bizonyítás:

$$\vec{u} = \operatorname{grad} f = \left(\frac{\partial f}{\partial x} ; \frac{\partial f}{\partial y} ; \frac{\partial f}{\partial z} \right) = \left(u_x ; u_y ; u_z \right)$$

$$\operatorname{div} \vec{u} = \vec{\nabla} \cdot \vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) =$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f$$

ITT A VÉGE, FUSS EL VÉLE …