1. $w = z^3 \quad \to \quad 3w^2 - 6w + 12 = 0 \quad \to \quad w^2 - 2w + 4 = 0$

$w_{1,2} = \frac{2\pm\sqrt{4-16}}{2} = 1 \pm \sqrt{3}i = 2(\cos(\pm 60°) + i\sin(\pm 60°))$

$$\boxed{\begin{aligned} z_{1,2,3} &= \sqrt[3]{w_1} = \sqrt[3]{2}(\cos(20° + k\cdot 120°) + i\sin(20° + k\cdot 120°)), \quad k = 0,1,2 \\ z_{4,5,6} &= \sqrt[3]{w_2} = \sqrt[3]{2}(\cos(-20° + k\cdot 120°) + i\sin(-20° + k\cdot 120°)), \quad k = 0,1,2 \end{aligned}}$$

2. $$a_{n+1} - a_n = \frac{\overbrace{2(n+1)+5}^{2n+7}}{\underbrace{4(n+1)-3}_{4n+1}} - \frac{2n+5}{4n-3} = \frac{(2n+7)(4n-3)-(2n+5)(4n+1)}{(4n+1)(4n-3)}$$

$$= \frac{-26}{\underbrace{(4n+1)}_{>0}\underbrace{(4n-3)}_{>0}} < 0 \quad \to \quad \boxed{\text{A sorozat szigorúan monoton csökkenő.}}$$

$$\lim_{n\to\infty} \frac{2n+5}{4n-3} = \lim_{n\to\infty} \frac{2+\frac{5}{n}}{4-\frac{3}{n}} = \boxed{\frac{1}{2} = A}$$

$$|a_n - A| < \varepsilon \quad \to \quad \left|\frac{2n+5}{4n-3} - \frac{1}{2}\right| < 10^{-4} \quad \to \quad \frac{|13|}{|4n-3|\cdot|2|} < 10^{-4} \quad \to$$

$$\frac{13}{(4n-3)\cdot 2} < 10^{-4} \quad \to \quad \frac{130000}{2} < 4n-3 \quad \to \quad n > \frac{65003}{4} \quad \to \quad \boxed{N = 16250}$$

3. $$\lim_{x\to\infty}\left(\sqrt{2x^2+3x+1} - \sqrt{2x^2-x+2}\right) = \lim_{x\to\infty} \frac{(2x^2+3x+1)-(2x^2-x+2)}{\sqrt{2x^2+3x+1}+\sqrt{2x^2-x+2}}$$

$$= \frac{4x-1}{\sqrt{2x^2+3x+1}+\sqrt{2x^2-x+2}} = \lim_{x\to\infty} \frac{4-\frac{1}{x}}{\sqrt{2+\frac{3}{x}+\frac{1}{x^2}}+\sqrt{2-\frac{1}{x}+\frac{2}{x^2}}} = \frac{4}{2\sqrt{2}} = \boxed{\sqrt{2}}$$

4. $$f(x_0) = \frac{\mathrm{sh}(0)}{\sqrt{4}} = 0, \quad f'(x) = \frac{\mathrm{ch}(2x+1)2\sqrt{4x+6} - \mathrm{sh}(2x+1)\frac{1}{2}(4x+6)^{-\frac{1}{2}}4}{4x+6}, \quad f'(x_0) = \frac{\mathrm{ch}(0)2\sqrt{4}}{4} = 1$$

Az érintő egyenlete: $y - f(x_0) = f'(x_0)(x - x_0) \quad \to \quad \boxed{y = x + \frac{1}{2}}$

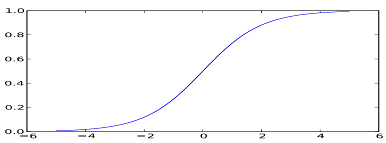
5. $D_f = \mathbb{R}$. Nem metszi az x tengelyt. Az y tengelyt $f(0) = \frac{1}{2}$-nél metszi.

$f(-1) \neq \pm f(1) \to$ nem páros, nem páratlan. $\lim\limits_{x\to\infty} f(x) = 1$, $\lim\limits_{x\to-\infty} f(x) = 0$.

$$f'(x) = \frac{e^{-x}}{(1+e^{-x})^2} > 0 \quad \to \quad \text{a függvény szigorúan monoton növekvő, lokális szélsőértéke nincs.}$$

$$f''(x) = \frac{e^{-x}(-1)(1+e^{-x})^2 - e^{-x}2(1+e^{-x})e^{-x}(-1)}{(1+e^{-x})^4} = \frac{e^{-x}(e^{-x}-1)}{(1+e^{-x})^3} = 0 \quad \to \quad x = 0$$

	$x<0$	$x=0$	$x>0$
f''	+	0	−
f	konvex		konkáv



$R_f = (0,1)$

6. (a) $$\int \frac{5}{16x^2+1}dx = 5\cdot\int\frac{1}{1+(4x)^2} = \boxed{\frac{5\mathrm{arctg}(4x)}{4} + C}$$

(b) $$\int_0^1 \underbrace{(1-2x)}_{f}\underbrace{\mathrm{sh}(3x)}_{g'}\,dx = \left[(1-2x)\frac{\mathrm{ch}(3x)}{3}\right]_0^1 - \int_0^1(-2)\frac{\mathrm{ch}(3x)}{3}dx$$

$$= \left[(1-2x)\frac{\mathrm{ch}(3x)}{3}\right]_0^1 + \frac{2}{3}\left[\frac{\mathrm{sh}(3x)}{3}\right]_0^1 = \boxed{\frac{-\mathrm{ch}(3)}{3} - \frac{1}{3} + \frac{2\mathrm{sh}(3)}{9}} \approx -1,463$$

(c) $$V = \pi\int_0^1\left(\frac{3}{\sqrt[3]{2x+3}}\right)^2 dx = 9\pi\int_0^1(2x+3)^{-\frac{2}{3}}dx = 9\pi\left[\frac{(2x+3)^{\frac{1}{3}}}{\frac{1}{3}2}\right]_0^1 = \boxed{\frac{27\pi}{2}\left(\sqrt[3]{5}-\sqrt[3]{3}\right)} \approx 11,35$$

7. (a) $y'x - 2yx^2 = x^2 \quad \rightarrow \quad y' \underbrace{-2x}_{a(x)} y = \underbrace{x}_{b(x)} \quad \rightarrow \quad y(x) = c(x)e^{-A(x)}$

$$c(x) = \int e^{A(x)} b(x) dx = \int e^{-x^2} x dx = -\frac{1}{2} \int e^{-x^2} (-2x) dx = -\frac{1}{2} e^{-x^2} + K$$

$$y(x) = \left(-\frac{1}{2} e^{-x^2} + K\right) e^{x^2} = \boxed{-\frac{1}{2} + K e^{x^2}}$$

$$y(1) = 2 \quad \rightarrow \quad -\frac{1}{2} + K e^{1^2} = 2 \quad \rightarrow \quad K = \frac{5}{2e} \quad \rightarrow \quad \boxed{y_p(x) = -\frac{1}{2} + \frac{5}{2e} e^{x^2}}$$

(b) $y_{(h,a)}: \; 4\lambda^2 + 8\lambda + 5 = 0 \quad \rightarrow \quad \lambda_{1,2} = \dfrac{-8 \pm \sqrt{64 - 80}}{8} = -1 \pm \dfrac{1}{2} i$

$y_{(i,p)}: \; y = Ax + B, \; y' = A, \; y'' = 0 \quad \rightarrow \quad 0 + 8A + 5(Ax + B) = 3x - 2 \quad \rightarrow \quad A = \dfrac{3}{5}, \; B = -\dfrac{34}{25}$

$$\boxed{y(x) = e^{-x} \left(C_1 \cos\left(\frac{1}{2} x\right) + C_2 \sin\left(\frac{1}{2} x\right)\right) + \frac{3}{5} x - \frac{34}{25}}$$