Nonhomogeneous linear differential equations

1. (Calculus 3, page 862, Exercise 57.) Solve the following equation using the method of undetermined coefficients:

$$y'' + 16y = e^{-2x}$$

Solution:
We follow the steps introduced on page 854. The the general solution to the nonhomogeneous equation is the sum of the general solution to the complementary equation and the particular solution:

$$y_{nonh} = y_h + y_p$$

(a) The complementary equation:

$$y'' + 16y = 0$$

The characteristic equation and its roots by the quadratic formula:

$$\lambda^2 + 16 = 0 \qquad \lambda_1 = 4i \quad \lambda_2 = -4i$$

Therefore the solution to the complementary equation:

$$y_h = c_1 \sin(4x) + c_2 \cos(4x)$$

(b) Since the right hand side function has the form e^{-2x}, our initial guess for the particular solution is $y_p = A \cdot e^{-2x}$.

(c) Since it is not a solution to the complementary equation, we substitute it into the differential equation. Before substituting, we determine the second derivative:

$$y_p = Ae^{-2x} \quad \rightarrow \quad y_p' = -2Ae^{-2x} \quad \rightarrow \quad y_p'' = 4Ae^{-2x}$$

Substitution:

$$4Ae^{-2x} + 16Ae^{-2x} = e^{-2x} \quad \rightarrow \quad A = 1/20 \quad \rightarrow \quad y_p = \frac{1}{20}e^{-2x}$$

(d) The general solution to the nonhomogeneous differental equation is the sum of the general solution to the complementary equation and the particular solution:

$$y_{nonh} = y_h + y_p = c_1 \sin(4x) + c_2 \cos(4x) + \frac{1}{20}e^{-2x}$$

2. (Calculus 3, page 862, Exercise 59.) Solve the following equation using the method of undetermined coefficients:

$$y'' - 4y' + 4y = 8x^2 + 4x$$

Solution:

We follow the steps introduced on page 854. The the general solution to the nonhomogeneous equation is the sum of the general solution to the complementary equation and the particular solution:

$$y_{nonh} = y_h + y_p$$

(a) The complementary equation:

$$y'' - 4y' + 4y = 0$$

The characteristic equation and its roots by the quadratic formula:

$$\lambda^2 - 4\lambda + 4 = 0 \qquad \lambda_1 = \lambda_2 = 2$$

Therefore the solution to the complementary equation:

$$y_h = c_1 e^{2x} + c_2 x e^{2x}$$

(b) Since the right hand side function is a polynomial of degree 2, our initial guess for the particular solution is $y_p = Ax^2 + Bx + C$.

(c) Since it is not a solution to the complementary equation, we substitute it into the differential equation. Before substituting, we determine the derivatives:

$$y_p = Ax^2 + Bx + C \quad \to \quad y_p' = 2Ax + B \quad \to \quad y_p'' = 2A$$

Substituting y_p, y_p' and y_p'' into the equation $y'' - 4y' + 4y = 8x^2 + 4x$:

$$2A - 4 \cdot (2Ax + B) + 4(Ax^2 + Bx + C) = 8x^2 + 4x$$

After rearranging the terms we have:

$$4Ax^2 + (4B - 8A)x + 4C - 4B + 2A = 8x^2 + 4x$$

Consequently, $4A = 8$, $4B - 8A = 4$ and $4C - 4B + 2A = 0$, so $A = 2$, $B = 5$ and $C = 4$. A particular solution is

$$y_p = 2x^2 + 5x + 4$$

(d) The general solution to the nonhomogeneous differental equation is the sum of the general solution to the complementary equation and the particular solution:

$$y_{nonh} = y_h + y_p = c_1 e^{2x} + c_2 x e^{2x} + 2x^2 + 5x + 4$$

3. (Calculus 3, page 862, Exercise 64.) Solve the following equation using the method of undetermined coefficients:

$$y'' + 3y' - 28y = 10e^{4x}$$

Solution:
The the general solution to the nonhomogeneous equation is the sum of the general solution to the complementary equation and the particular solution:

$$y_{nonh} = y_h + y_p$$

(a) The complementary equation:

$$y'' + 3y' - 28y = 0$$

The characteristic equation and its roots by the quadratic formula:

$$\lambda^2 + 3\lambda - 28 = 0 \qquad \lambda_1 = -7 \quad \lambda_2 = 4$$

Therefore the solution to the complementary equation:

$$y_h = c_1 e^{-7x} + c_2 e^{4x}$$

(b) Since the right hand side function is $10e^{4x}$, our initial guess for the particular solution is $y_p = Ae^{4x}$.

(c) But e^{4x} is a solution to the complementary equation, so we multiply it by x: $y_p = Axe^{4x}$. This new y_p is not a solution to the complementary equation. Determine the derivatives:

$$y_p = Axe^{4x} \quad \to \quad y_p' = Ae^{4x}(1 + 4x) \quad \to \quad y_p'' = Ae^{4x}(16x + 8)$$

Substituting y_p, y_p' and y_p'' into the equation $y'' + 3y' - 28y = 10e^{4x}$:

$$Ae^{4x}(16x + 8) + 3Ae^{4x}(1 + 4x) - 28Axe^{4x} = 10e^{4x}$$

After simplifying by e^{4x} we have:

$$\begin{aligned} A(16x + 8) + 3A(1 + 4x) - 28Ax &= 10 \\ x(16A + 12A - 28A) + 8A + 3A &= 10 \\ A &= \frac{10}{11} \end{aligned}$$

Consequently, $y_p = \frac{10}{11}xe^{4x}$.

(d) The general solution to the nonhomogeneous differental equation is the sum of the general solution to the complementary equation and the particular solution:

$$y_{nonh} = y_h + y_p = c_1 e^{-7x} + c_2 e^{4x} + \frac{10}{11}xe^{4x}$$

4. (Calculus 3, page 862, Exercise 80.) Find the unique solution satisfying the differential equation and the initial conditions given, where y_p is the particular solution:

$$y'' - 2y' + y = 12e^x, \quad y(0) = 6, \quad y'(0) = 0, \quad y_p = 6x^2e^x$$

Solution:
First we have to find the general solution to the nonhomogeneous equation, which is the sum of the general solution to the complementary equation and the particular solution:

$$y_{nonh} = y_h + y_p$$

The complementary equation:

$$y'' - 2y' + y = 0$$

The characteristic equation and its roots:

$$\lambda^2 - 2\lambda + 1 = 0 \qquad \lambda_1 = \lambda_2 = 1$$

Therefore the solution to the complementary equation is:

$$y_h = c_1e^x + c_2xe^x$$

The general solution to the nonhomogeneous equation is:

$$y_{nonh} = y_h + y_p = c_1e^x + c_2xe^x + 6x^2e^x$$

Now let's use the initial conditions, but first we compute the derivative:

$$y'_{nonh} = c_1e^x + c_2(x+1)e^x + (6x^2 + 12x)e^x$$

Using the initial conditions:

$$y(0) = c_1 = 6$$
$$y'(0) = c_1 + c_2 = 0$$

So $c_1 = 6$ and $c_2 = -6$. Therefore the solution to the initail value problem is the following function:

$$y = 6e^x - 6xe^x + 6x^2e^x$$

Or in other form:

$$y = 6e^x(1 - x + x^2)$$

5. (Calculus 3, page 879 , initial value problem from the student project) Solve the initial value problem:

$$x'' + x = 5\cos t \qquad x(0) = 0, \quad x'(0) = 1$$

Solution:
First we have to find the general solution to the nonhomogeneous equation:

$$x_{nonh} = x_h + x_p$$

The complementary equation:

$$x'' + x = 0$$

The characteristic equation and its roots:

$$\lambda^2 + 1 = 0 \qquad \lambda_1 = i \quad \lambda_2 = -i$$

Therefore the solution to the complementary equation is:

$$y_h = c_1 \sin t + c_2 \cos t$$

Since the right hand side function is $5\cos t$, our initial guess for the particular solution is $x_p = A\cos t + B\sin t$. But $\cos t$ is a solution to the complementary equation, so we multiply it by t: $x_p = At\cos t + Bt\sin t$. Its derivatives:

$$\begin{aligned} x_p' &= (At\cos t + Bt\sin t)' = A\cos t - At\sin t + B\sin t + Bt\cos t \\ x_p'' &= (A\cos t - At\sin t + B\sin t + Bt\cos t)' = -2A\sin t - At\cos t + 2B\cos t - Bt\sin t \end{aligned}$$

Substituting into the equation $x'' + x = 5\cos t$:

$$\begin{aligned} \overbrace{-2A\sin t - At\cos t + 2B\cos t - Bt\sin t}^{x_p''} + \overbrace{At\cos t + Bt\sin t}^{x_p} &= 5\cos t \\ -2A\sin t + 2B\cos t &= 5\cos t \end{aligned}$$

Consequently, $A = 0$ and $B = 5/2$, and the particular solution is $x_p = \frac{5}{2}t\sin t$. So the general solution to the nonhomogeneous equation is

$$x(t) = c_1 \sin t + c_2 \cos t + \frac{5}{2}t\sin t$$

Now we determine the values of c_1 and c_2. The derivative is

$$x' = c_1 \cos t - c_2 \sin t + \frac{5}{2}\sin t - \frac{5}{2}t\cos t$$

Applying the initial conditions:

$$\begin{aligned} x(0) &= c_2 = 0 \\ x'(0) &= c_1 = 1 \end{aligned}$$

The solution to the initial value problem is

$$x(t) = \sin t + \frac{5}{2}t\sin t = \left(1 + \frac{5}{2}t\right)\sin t$$

6. (Calculus 3, page 878, Exercise 7.20) A mass of 2 kg is attached to a spring with constant 32 N/m and comes to rest in the equilibrium position. Beginning at time $t = 0$, an external force equal to $f(t) = 68e^{-2t}\cos(4t)$ is applied to the system. Find the equation of motion if there is no damping. What is the transient solution? What is the steady-state solution?

Solution:
Let's denote the position at time t by $x(t)$. Then according to Newton's second law (and assuming Hooke's law):

$$mx''(t) + kx(t) = f(t)$$

It was given that $m = 2\,\mathrm{kg}$ and $k = 32\,\mathrm{N/m}$. Moreover, form the text $x(0) = 0$ (it starts from the equlibrium point) and $x'(0) = 0$ (it rests in the equlibrium point). Finally we arrive to the following initial value problem:

$$2x'' + 32x = 68e^{-2t}\cos(4t) \qquad x(0) = 0, \quad x'(0) = 0$$

First we determine the general solution to the complementary (homogeneous) equation (we simplified by 2):

$$x'' + 16x = 0$$

The corresponding characteristic equation and its roots:

$$\lambda^2 + 16 = 0 \qquad \lambda_1 = 4i, \quad \lambda_2 = -4i$$

The general solution to the complementary equation:

$$x_h = c_1 \sin(4t) + c_2 \cos(4t)$$

The external force is given in the form $f(t) = 68e^{-2t}\cos(4t)$, so we search for the particular solution in the form of $e^{-2t}(A\sin(4t) + B\cos(4t))$. Neither $e^{-2t}\sin(4t)$ or $e^{-2t}B\cos(4t))$ is contained in the solution of the homogeneous equation, so we keep this form. After combining terms:

$$\begin{aligned}
x_p &= e^{-2t}\left[A\sin(4t) + B\cos(4t)\right] \\
x_p' &= e^{-2t}\left[(4A - 2B)\cos(4t) - (2A + 4B)\sin(4t)\right] \\
x_p'' &= e^{-2t}\left[(-12A + 16B)\sin(4t) - (16A + 12B)\cos(4t)\right]
\end{aligned}$$

Substituting into the equation $x'' + 16x = 34e^{-2t}\cos(4t)$:

$$\begin{aligned}
e^{-2t}\left[(-12A + 16B)\sin(4t) - (16A + 12B)\cos(4t)\right] + 16 \cdot e^{-2t}\left[A\sin(4t) + B\cos(4t)\right] &= 34e^{-2t}\cos(4t) \\
(-12A + 16B + 16A)\sin(4t) + (-16A - 12B + 16B)\cos(4t) &= 34\cos(4t)
\end{aligned}$$

So we have:

$$\begin{aligned}
-12A + 16B + 16A = 4A + 16B = 0 \\
-16A - 12B + 16B = -16A + 4B = 34
\end{aligned}$$

The solution to this system of equations is $A = -2$ and $B = 1/2$. Consequently, the particular solution is

$$x_p(t) = e^{-2t}\left[-2\sin(4t) + \frac{1}{2}\cos(4t)\right]$$

The general solution to the nonhomogeneous equation is

$$x_{nonh} = x_h + x_p = c_1 \sin(4t) + c_2 \cos(4t) + e^{-2t} \left[-2\sin(4t) + \frac{1}{2}\cos(4t) \right]$$

Let's determine the value of c_1 and c_2 using the initial conditions. For this task we first compute the derivate of x_{nonh}:

$$x'_{nonh} = 4c_1 \cos(4t) - 4c_2 \sin(4t) - 2e^{-2t} \left[-2\sin(4t) + \frac{1}{2}\cos(4t) \right] + e^{-2t} \left[-8\cos(4t) - 2\sin(4t) \right]$$

Applying the initial conditions:

$$x(0) = c_2 + \frac{1}{2} = 0$$
$$x'(0) = 4c_1 - 1 + -8 = 0$$

Then $c_2 = -1/2$ and $c_1 = 9/4$. The solution to the initial value problem is

$$x(t) = \frac{9}{4}\sin(4t) - \frac{1}{2}\cos(4t) + e^{-2t} \left[-2\sin(4t) + \frac{1}{2}\cos(4t) \right]$$

What happens if $t \to \infty$? The exponential term e^{-2t} goes to zero, the trigonometrical terms are bounded functions, so the second part of the solution converges to zero.

Transient solution:

$$e^{-2t} \left[-2\sin(4t) + \frac{1}{2}\cos(4t) \right]$$

Steady-state solution:

$$\frac{9}{4}\sin(4t) - \frac{1}{2}\cos(4t)$$

7. (Calculus 3, page 882, Exercise 94.) A 9-kg mass is attached to a vertical spring with a spring constant of 16 N/m. The system is immersed in a medium that imparts a damping force equal to 24 times the instantaneous velocity of the mass.

   (a) Find the equation of motion if it is released from its equilibrium position with an upward velocity of 4 m/sec.

   (b) Graph the solution and determine whether the motion is overdamped, critically damped, or underdamped.

   Solution:

   (a) Let's denote the position at time t by $x(t)$. Then according to Newton's second law:

$$mx''(t) + bx'(t) + kx(t) = 0$$

   where $m = 9\,\text{kg}$, $b = 24\,\text{kg/s}$ and $k = 16\,\text{N/m}$. The initial conditions are $x(0) = 0$, since we start from the equilibrium point; $x'(0) = -4$, since the upward direction is the negative (the downward direction is the direction of the strain, so it is considered as positive). So we have to solve the following initial value problem:

$$9x'' + 24x' + 16x = 0 \qquad x(0) = 0, \quad x'(0) = -4$$

   The corresponding characteristic equation and its solutions are

$$9\lambda^2 + 24\lambda + 16 = 0 \qquad \lambda_1 = \lambda_2 = -\frac{4}{3}$$

   The general solution is

$$x_h = c_1 e^{-\frac{4}{3}x} + c_2 x e^{-\frac{4}{3}x}$$

   The derivative is

$$x_h' = -\frac{4}{3}c_1 e^{-\frac{4}{3}x} + c_2 e^{-\frac{4}{3}x} - \frac{4}{3}c_2 x e^{-\frac{4}{3}x}$$

   Applying the initial conditions:

$$x(0) = c_1 = 0$$
$$x'(0) = -\frac{4}{3}c_1 + c_2 = -4$$

   Then $c_1 = 0$ and $c_2 = -4$. The solution to the initial value problems is

$$x(t) = -4xe^{-\frac{4}{3}x}$$

   (b) We have the examine the sign of $b^2 - 4mk$. In our case $b = 24$, $m = 8$ and $k = 16$, so $b^2 - 4mk = 24^2 - 4 \cdot 9 \cdot 16 = 0$. The system is critically damped. See the graph on the next page.