

4. KONZULTÁCIÓ

CASTIGLIANO TÉTEL

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- A szerkezetet terhelő F_i erő P_i támadáspontjának F_i irányba eső elmozdulása egyenlő a szerkezet belső energiájának F_i erő szerinti parciális deriváltjával

$$u_i = \frac{\partial U}{\partial F_{ix}} \quad v_i = \frac{\partial U}{\partial F_{iy}} \quad w_i = \frac{\partial U}{\partial F_{iz}}$$

- A szerkezetet terhelő M_i nyomaték P_i támadáspontjában lévő keresztmetszet M_i nyomaték tengelye körüli szögelfordulása egyenlő a szerkezet belső energiájának az M_i nyomaték szerinti parciális deriváltjával

$$\varphi_{ix} = \frac{\partial U}{\partial M_{ix}} \quad \varphi_{iy} = \frac{\partial U}{\partial M_{iy}} \quad \varphi_{iz} = \frac{\partial U}{\partial M_{iz}}$$

- A szerkezet belső energiája (alakváltozás energia):

$$U = \frac{1}{2} \int_{(l)} \frac{N^2}{AE} ds + \frac{1}{2} \int_{(l)} \frac{M_{hx}^2}{I_x E} ds + \frac{1}{2} \int_{(l)} \frac{M_{hy}^2}{I_y E} ds + \frac{1}{2} \int_{(l)} \frac{M_c^2}{I_p G} ds$$

- Felhasználás:

- elmozdulás/szögelfordulás számítás
- statikailag határozatlan szerkezet támasztóerői

INTEGRÁLÁSI SZABÁLYOK

• Addíciós tételek:

$$1) \sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$2) \cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\bullet \text{ Pitagorasz tétel} \rightarrow \cos^2 f + \sin^2 f = 1$$

$$\bullet 2) \text{ Addíciós tétel} \rightarrow \cos^2 f - \sin^2 f = \cos 2f$$

$$\oplus \quad 2 \cos^2 f = 1 + \cos 2f$$

$$\boxed{\cos^2 f = \frac{1 + \cos 2f}{2}}$$

$$\ominus \quad 2 \sin^2 f = 1 - \cos 2f$$

$$\boxed{\sin^2 f = \frac{1 - \cos 2f}{2}}$$

$$\bullet 1) \text{ Addíciós tétel} \rightarrow \sin 2f = 2 \sin f \cos f$$

$$\boxed{\sin f \cos f = \frac{\sin 2f}{2}}$$

• trigonometrikus
függvények
integrálása:

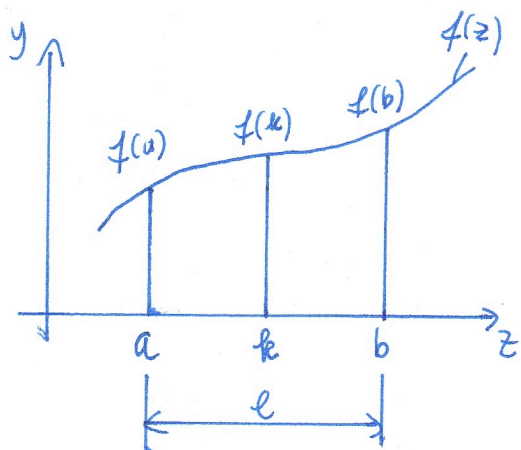
$$\int \sin f = -\cos f$$

$$\int \cos f = \sin f$$

$$\int \sin(Af) = -\frac{\cos(Af)}{A}$$

$$\int \cos(Af) = \frac{\sin(Af)}{A}$$

• SIMPSON FORMULA

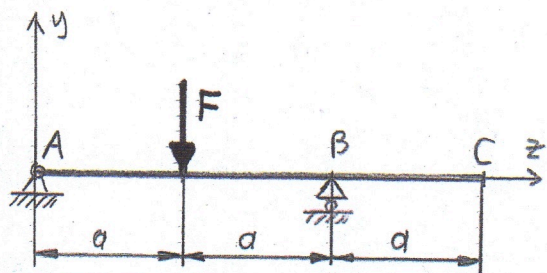


$$\int_a^b f(z) dz = \frac{l}{6} (f(a) + 4f(k) + f(b))$$

$$k = \frac{a+b}{2} \quad l = b-a$$

- 3-ad fokú polinomig exakt megoldást ad
- töreks esetén szakaszonként alkalmazandó

4.1 - EGYENES RÚD ALAKVÁLTOZÁSA



adott:

$$F = 4 \text{ kN}$$

$$a = 2 \text{ m}$$

$$E = 2 \cdot 10^5 \text{ MPa}$$

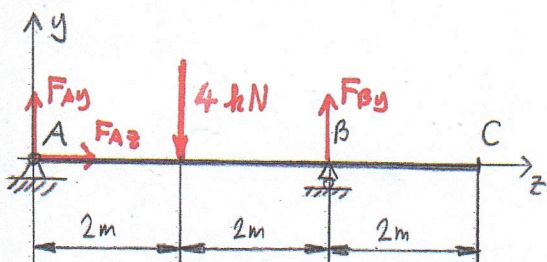
$$I_x = 2,5 \cdot 10^5 \text{ mm}^4$$

Feladat:

C keresztmetszet y irányú elmozdulása

$$v_c = ?$$

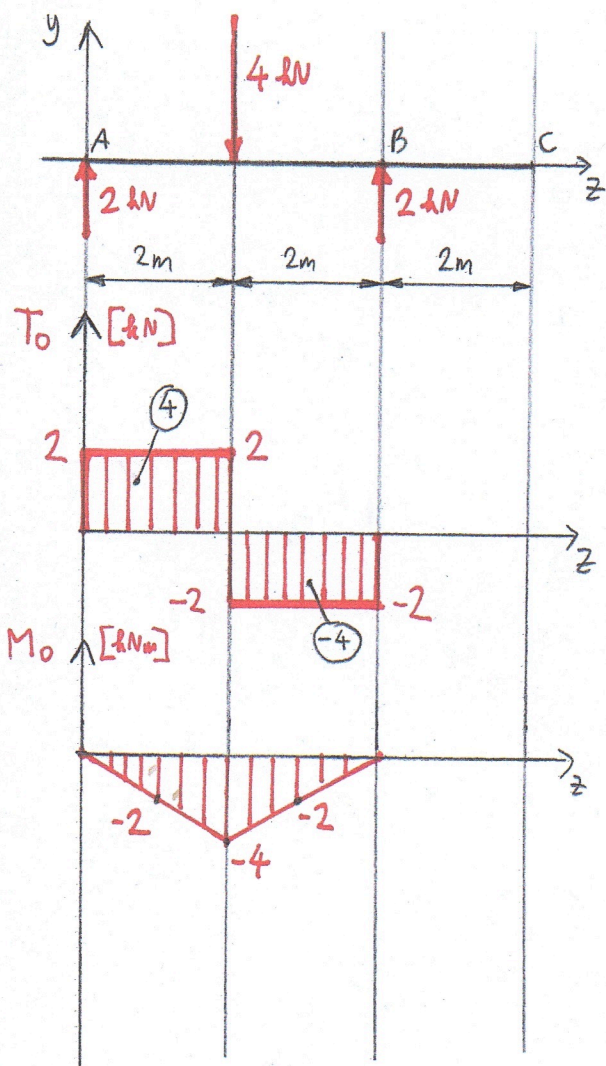
• eredeti terhelés



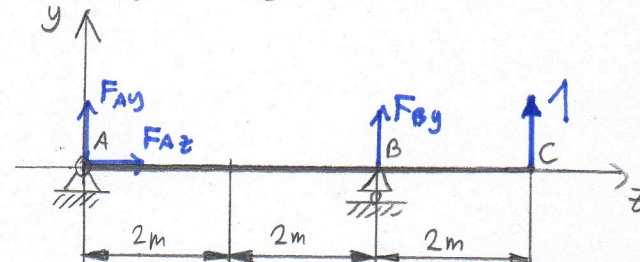
$$M_a = 0 = 2 \cdot 4 - 4 F_{By} \Rightarrow F_{By} = 2 \text{ kN} (\uparrow)$$

$$M_b = 0 = -2 \cdot 4 + 4 F_{Ay} \Rightarrow F_{Ay} = 2 \text{ kN} (\uparrow)$$

$$F_z = 0 = F_{Az}$$



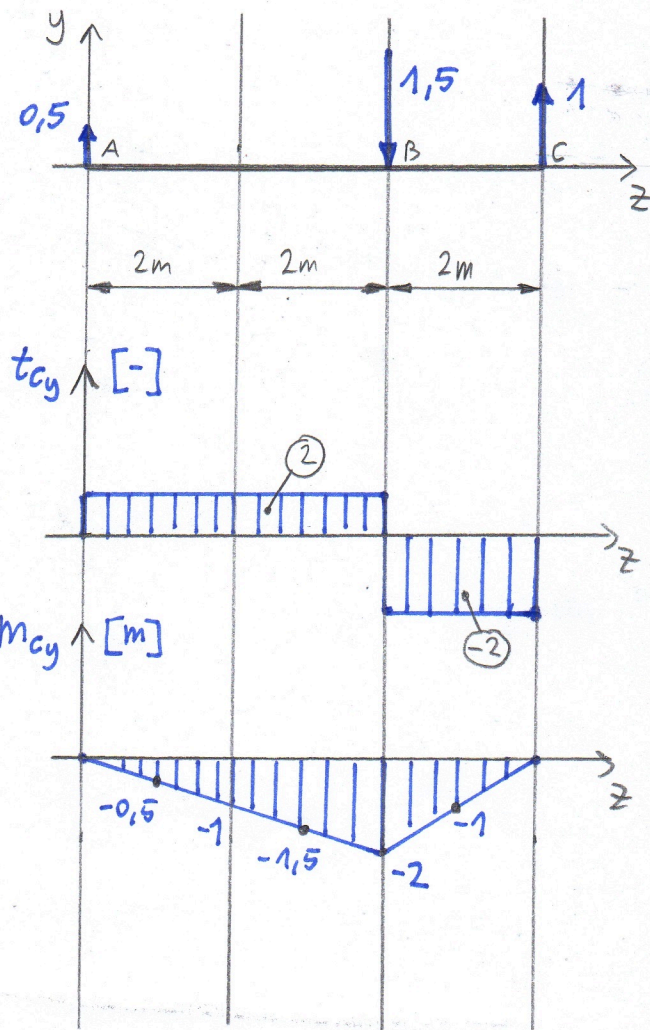
• egységnyi F_{Cy} terhelés



$$M_a = 0 = -4 F_{By} - 6 \cdot 1 \Rightarrow F_{By} = -1,5 (\downarrow)$$

$$M_b = 0 = 4 F_{Ay} - 2 \cdot 1 \Rightarrow F_{Ay} = 0,5 (\uparrow)$$

$$F_z = 0 = F_{Az}$$



• az alakváltozási energia:

$$U \approx U_H = \frac{1}{2} \int_0^{3d} \frac{M_{hx}^2}{I_x E} dz = \frac{1}{2} \int_0^{3d} \frac{(M_0 + F_{cy} m_{cy})^2}{I_x E} dz$$

• Castigliano - tétel + dinamikai peremfeltétel:

$$V_c = \left. \frac{\partial U}{\partial F_{cy}} \right|_{F_{cy}=0}$$

$$V_c = \left. \int_0^{3d} \frac{(M_0 + F_{cy} m_{cy}) m_{cy}}{I_x E} dz \right|_{F_{cy}=0} = \frac{1}{I_x E} \int_0^{3d} M_0 m_{cy} dz$$

• integrál kiszámítása Simpson formulával:

$$\int_0^{3d} M_0 m_{cy} dz = \int_0^d M_0 m_{cy} dz + \int_d^{2d} M_0 m_{cy} dz + \underbrace{\int_{2d}^{3d} M_0 m_{cy} dz}_{=0}$$

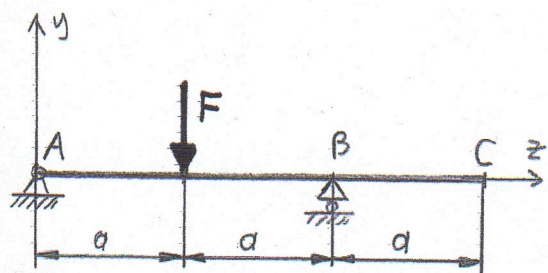
$$= \frac{2}{6} \left[0 \cdot 0 + 4 \cdot (-2) \cdot (-0,5) + (-4) \cdot (-1) \right] + \frac{2}{6} \left[(-4) \cdot (-1) + 4 \cdot (-2) \cdot (-1,5) + 0 \cdot (-2) \right] =$$

$$= \frac{2}{6} [18] + \frac{2}{6} [16] = \frac{2}{6} [8+16] = \frac{48}{6} = 8 \text{ Nm}^3 = 8 \cdot 10^{12} \text{ Nmm}^3$$

• behelyettesítve V_c képletébe:

$$V_c = \frac{1}{I_x E} \cdot (8 \cdot 10^{12}) = \frac{8 \cdot 10^{12}}{2,5 \cdot 10^5 \cdot 2 \cdot 10^5} = \underline{\underline{160 \text{ mm} (\uparrow)}}$$

4.2 - EGYENES RÚD ALAKVÁLTOZÁSA



adott:

$$F = 4 \text{ kN}$$

$$a = 2 \text{ m}$$

$$E = 2 \cdot 10^5 \text{ MPa}$$

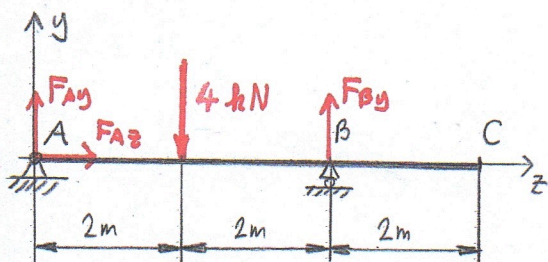
$$I_x = 2,5 \cdot 10^5 \text{ mm}^4$$

Feladat:

C keresztmetszet x tengely
Körli szögelfordulása

$$\varphi_{Cx} = ?$$

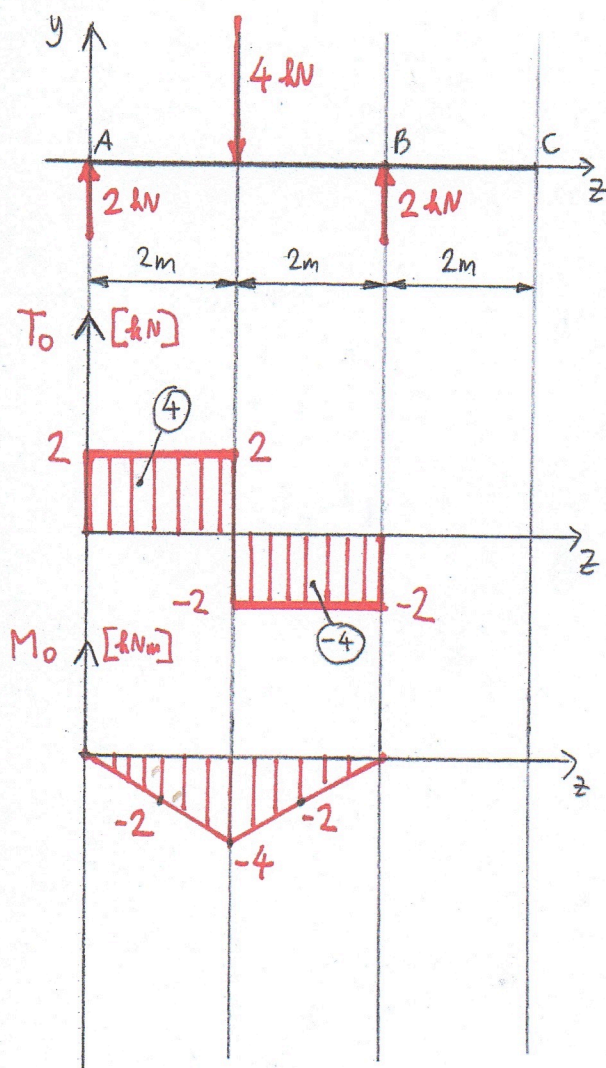
• eredeti terhelés



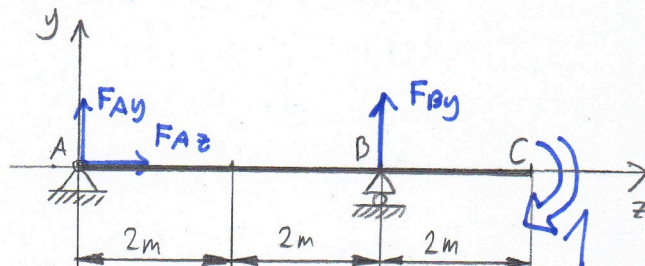
$$M_a = 0 = 2 \cdot 4 - 4 F_{By} \Rightarrow F_{By} = 2 \text{ kN} (\uparrow)$$

$$M_b = 0 = -2 \cdot 4 + 4 F_{Ay} \Rightarrow F_{Ay} = 2 \text{ kN} (\uparrow)$$

$$F_z = 0 = F_{Az}$$



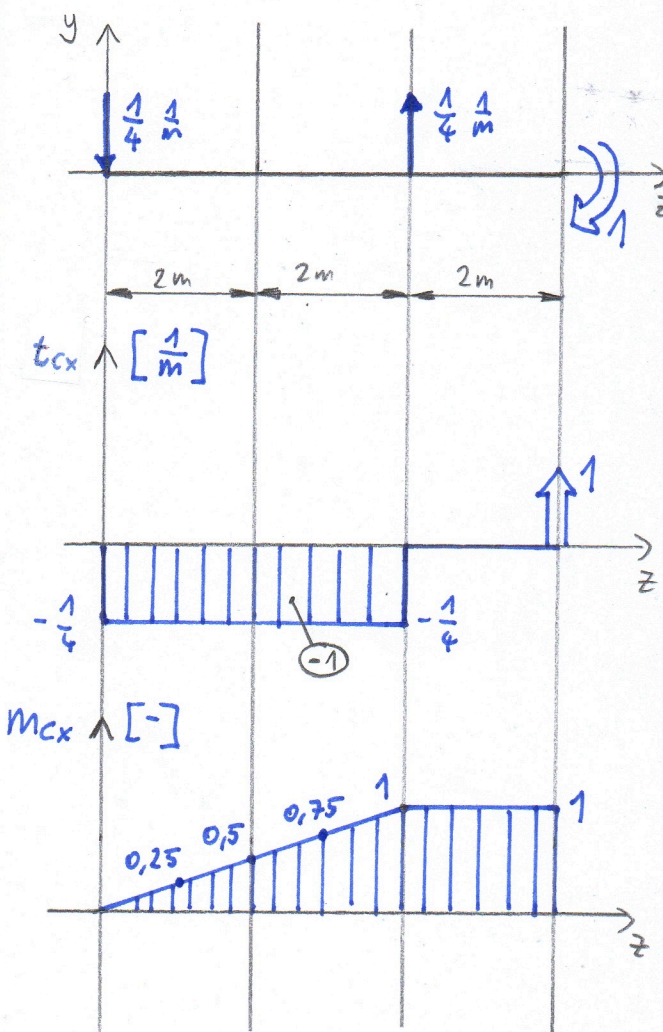
• egységnyi M_{Cx} terhelés



$$M_a = 0 = -4 F_{By} + 1 \Rightarrow F_{By} = \frac{1}{4} \frac{1}{\text{m}} (\uparrow)$$

$$M_b = 0 = 4 F_{Ay} + 1 \Rightarrow F_{Ay} = -\frac{1}{4} \frac{1}{\text{m}} (\downarrow)$$

$$F_z = 0 = F_{Az}$$



• az alakváltozási energia:

$$U \approx U_H = \frac{1}{2} \int_0^{3a} \frac{M_{hx}^2}{I_x E} dz = \frac{1}{2} \int_0^{3a} \frac{(M_0 + M_{cx} m_{cx})^2}{I_x E} dz$$

• Castigliano-tétel + dinamikai peremfeltétel:

$$\varphi_{cx} = \left. \frac{\partial U}{\partial M_{cx}} \right|_{M_{cx}=0}$$

$$\varphi_{cx} = \left. \int_0^{3a} \frac{(M_0 + M_{cx} m_{cx}) m_{cx}}{I_x E} dz \right|_{M_{cx}=0} = \frac{1}{I_x E} \int_0^{3a} M_0 m_{cx} dz$$

• integrál kiszámítása Simpson formulával:

$$\int_0^{3a} M_0 m_{cx} dz = \int_0^d M_0 m_{cx} dz + \int_d^{2d} M_0 m_{cx} dz + \underbrace{\int_{2a}^{3a} M_0 m_{cx} dz}_{=0} =$$

$$= \frac{2}{6} [0 \cdot 0 + 4 \cdot (-2)(0,25) + (-4) \cdot 0,5] + \frac{2}{6} [(-4) \cdot 0,5 + 4 \cdot (-2) \cdot 0,75 + 0 \cdot 1] =$$

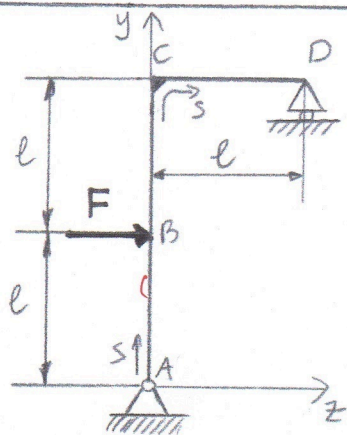
$$= \frac{2}{6} [-4] + \frac{2}{6} [-8] = \frac{2}{6} [-4-8] = -\frac{24}{6} = -4 \text{ kNm}^2 = -4 \cdot 10^9 \text{ Nmm}^2$$

• behelyettesítve φ_{cx} képletébe:

$$\varphi_{cx} = \frac{1}{I_x E} \cdot (-4 \cdot 10^9) = \frac{-4 \cdot 10^9}{2,3 \cdot 10^5 \cdot 2 \cdot 10^5} = -0,08 \text{ rad } \S$$

$$\varphi_{cx} = -0,08 \cdot \frac{180}{\pi} = \underline{\underline{-4,58^\circ \S}}$$

4.3 - TÖRTVONALÚ TARTÓ ALAKVÁLTOZÁSA



adott:

$$l = 500 \text{ mm}$$

$$d = 40 \text{ mm}$$

$$E = 2 \cdot 10^5 \text{ MPa}$$

$$F = 3000 \text{ N}$$

Feladat:

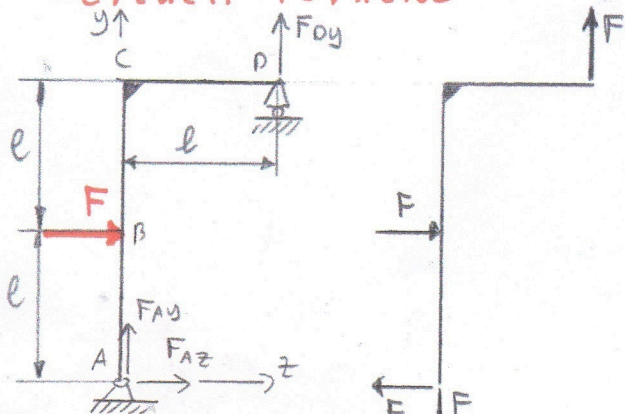
D keresztmetszet súlypontjának
z irányú elmozdulása

$$u_D = ?$$

$$A = \frac{d^2 \pi}{4} = \frac{40^2 \pi}{4} = 1256,637 \text{ mm}^2$$

$$I_x = \frac{d^4 \pi}{64} = \frac{40^4 \pi}{64} = 125663,7 \text{ mm}^4$$

• eredeti terhelés

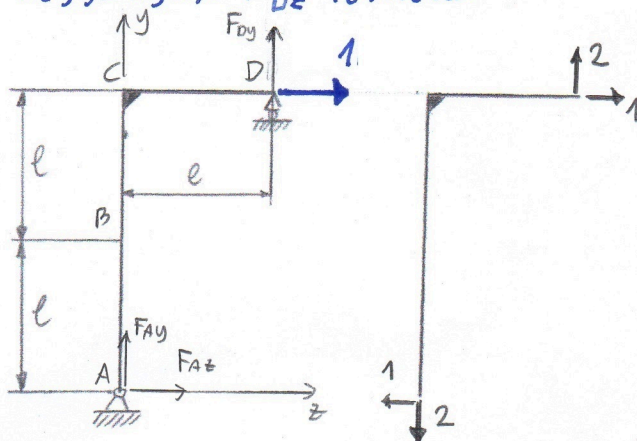


$$M_D = 0 = lF - lF_{Dy} \Rightarrow F_{Dy} = F (\uparrow)$$

$$F_y = 0 = F_{Ay} + F_{Dy} \Rightarrow F_{Ay} = -F_{Dy} = -F (\downarrow)$$

$$F_z = 0 = F_{Az} + F \Rightarrow F_{Az} = -F (\leftarrow)$$

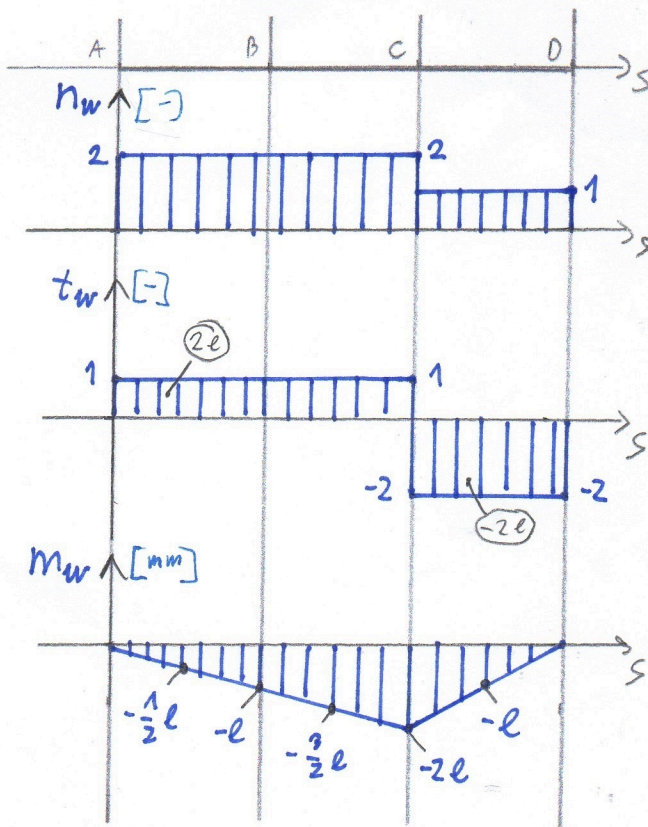
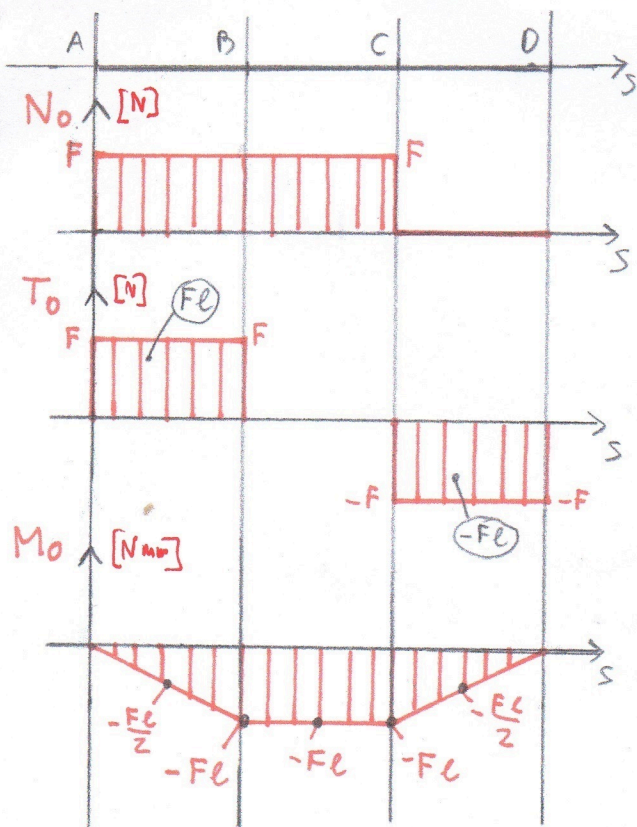
• egységnyi F_{Dz} terhelés



$$M_D = 0 = 2l - lF_{Dy} \Rightarrow F_{Dy} = 2 (\uparrow)$$

$$F_y = 0 = F_{Ay} + F_{Dy} \Rightarrow F_{Ay} = -F_{Dy} = -2 (\downarrow)$$

$$F_z = 0 = F_{Az} + 1 \Rightarrow F_{Az} = -1 \text{ N} (\leftarrow)$$



• az alakváltozási energia:

$$U \approx U_N + U_H = \frac{1}{2} \int_0^{3l} \frac{N^2}{AE} ds + \frac{1}{2} \int_0^{3l} \frac{M_{hx}^2}{I_x E} ds = \frac{1}{2} \int_0^{3l} \frac{(N_0 + F_{Dz} n_w)^2}{AE} ds + \frac{1}{2} \int_0^{3l} \frac{(M_0 + F_{Dz} m_w)^2}{I_x E} ds$$

• Castigliano-tétel:

$$w_D = \left. \frac{\partial U}{\partial F_{Dz}} \right|_{F_{Dz}=0}$$

$$w_D = \left. \int_0^{3l} \frac{(N_0 + F_{Dz} n_w) n_w}{AE} ds \right|_{F_{Dz}=0} + \left. \int_0^{3l} \frac{(M_0 + F_{Dz} m_w) m_w}{I_x E} ds \right|_{F_{Dz}=0}$$

$$w_D = \frac{1}{AE} \int_0^{3l} N_0 n_w ds + \frac{1}{I_x E} \int_0^{3l} M_0 m_w ds$$

• integrálok kiszámítása Simpson formulával:

$$\int_0^{3l} N_0 n_w ds = \int_0^{2l} N_0 n_w ds + \underbrace{\int_{2l}^{3l} N_0 n_w ds}_{=0} = \frac{2l}{6} \left[\underbrace{F \cdot 2 + 4 \cdot F \cdot 2 + F \cdot 2}_{12F} \right] = 4lF$$

$$\begin{aligned} \int_0^{3l} M_0 m_w ds &= \int_0^l M_0 m_w ds + \int_l^{2l} M_0 m_w ds + \int_{2l}^{3l} M_0 m_w ds = \\ &= \frac{l}{6} \left[0 \cdot 0 + 4 \cdot \left(-\frac{Fl}{2} \right) \left(-\frac{1}{2}l \right) + (-Fl)(-l) \right] + \frac{l}{6} \left[(-Fl)(-l) + 4 \cdot (-Fl) \left(-\frac{3}{2}l \right) + (-Fl)(-2l) \right] + \frac{l}{6} \left[(-Fl)(-2l) + 4 \cdot \left(-\frac{Fl}{2} \right) (-l) + 0 \right] = \\ &= \frac{l}{6} \left[2Fl^2 + 9Fl^2 + 4Fl^2 \right] = \frac{l}{6} \cdot 15Fl^2 = \frac{15Fl^3}{6} = \frac{5Fl^3}{2} \end{aligned}$$

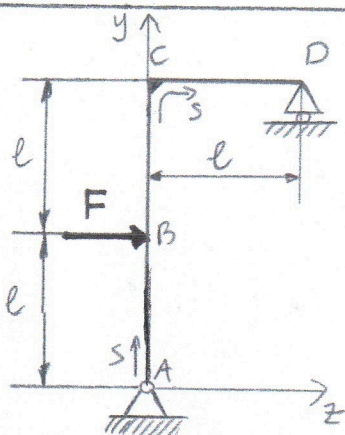
• keresztmetszeti jellemzők:

$$A = \frac{d^2 \pi}{4} = \frac{40^2 \pi}{4} = 1256,637 \text{ mm}^2; I_x = \frac{d^4 \pi}{64} = \frac{40^4 \pi}{64} = 125663,7 \text{ mm}^4$$

• behelyettesítve w_D képletébe:

$$\begin{aligned} w_D &= \frac{1}{AE} (4lF) + \frac{1}{I_x E} \left(\frac{5Fl^3}{2} \right) = \frac{4lF}{AE} + \frac{5Fl^3}{2I_x E} = \\ &= \frac{4 \cdot 500 \cdot 3000}{1256,637 \cdot 2 \cdot 10^5} + \frac{5 \cdot 3000 \cdot 500^3}{2 \cdot 125663,7 \cdot 2 \cdot 10^5} = \underline{\underline{37,33 \text{ mm} (\rightarrow)}} \end{aligned}$$

4.4 - TÖRTVONALÚ TARTÓ ALAKVÁLTOZÁSA



adott:

$$l = 500 \text{ mm}$$

$$d = 40 \text{ mm}$$

$$E = 2 \cdot 10^5 \text{ MPa}$$

$$F = 3000 \text{ N}$$

Feladat:

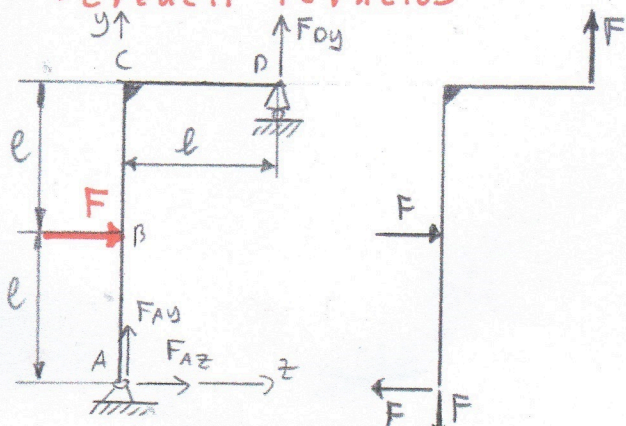
A keresztmetszet x tengely körüli
szögelfordulása

$$\varphi_A = ?$$

$$A = \frac{d^2 \pi}{4} = \frac{40^2 \pi}{4} = 1256,637 \text{ mm}^2$$

$$I_x = \frac{d^4 \pi}{64} = \frac{40^4 \pi}{64} = 125663,7 \text{ mm}^4$$

• eredeti terhelés

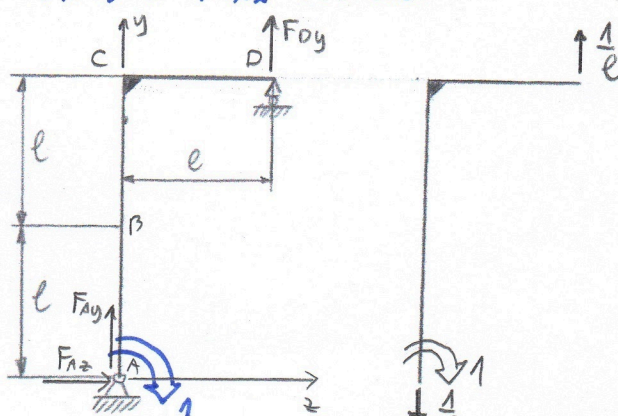


$$M_D = 0 = lF - lF_{Dy} \Rightarrow F_{Dy} = F (\uparrow)$$

$$F_y = 0 = F_{Ay} + F_{Dy} \Rightarrow F_{Ay} = -F_{Dy} = -F (\downarrow)$$

$$F_z = 0 = F_{Az} + F \Rightarrow F_{Az} = -F (\leftarrow)$$

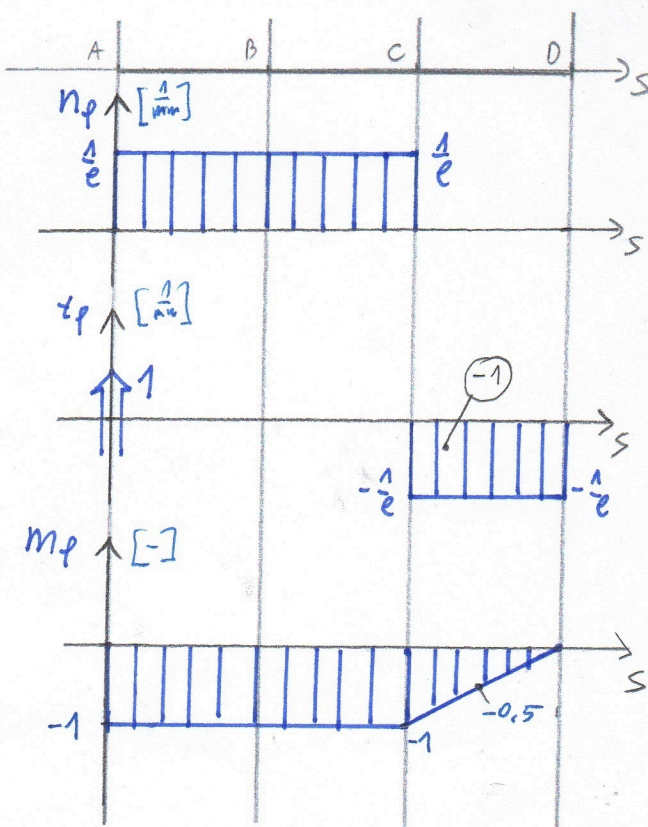
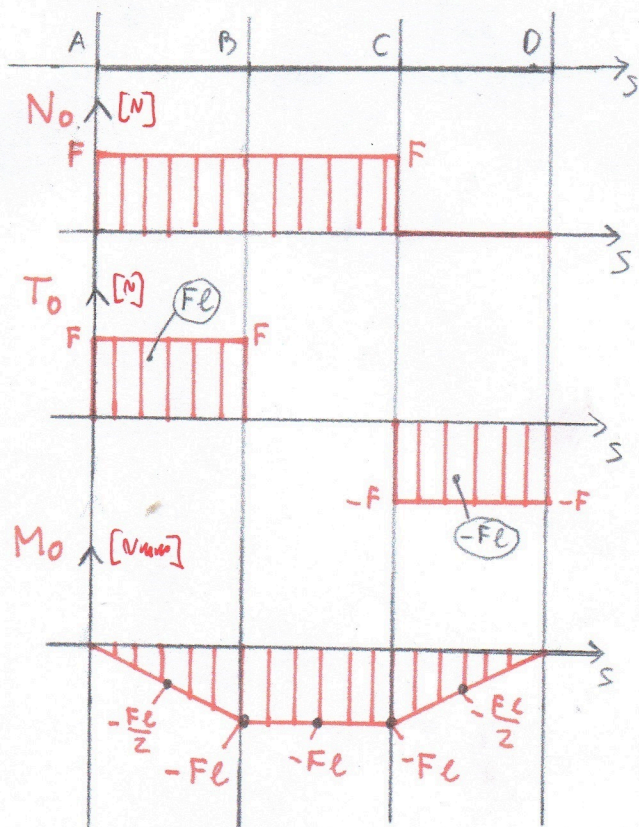
• egyenértékű Max terhelés



$$M_D = 0 = 1 - lF_{Dy} \Rightarrow F_{Dy} = \frac{1}{l}$$

$$F_y = 0 = F_{Ay} + F_{Dy} \Rightarrow F_{Ay} = -F_{Dy} = -\frac{1}{l}$$

$$F_z = 0 = F_{Az}$$



• az alakváltozási energia:

$$U \approx U_N + U_H = \frac{1}{2} \int_0^{3l} \frac{N^2}{AE} ds + \frac{1}{2} \int_0^{3l} \frac{M_{hx}^2}{I_x E} ds = \frac{1}{2} \int_0^{3l} \frac{(N_0 + M_{Ax} n_f)^2}{AE} ds + \frac{1}{2} \int_0^{3l} \frac{(M_0 + M_{Ax} m_f)^2}{I_x E} ds$$

• Castigliano-tétel:

$$f_A = \left. \frac{\partial U}{\partial M_{Ax}} \right|_{M_{Ax}=0}$$

$$f_A = \int_0^{3l} \frac{(N_0 + M_{Ax} n_f) n_f}{AE} ds \Big|_{M_{Ax}=0} + \int_0^{3l} \frac{(M_0 + M_{Ax} m_f) m_f}{I_x E} ds \Big|_{M_{Ax}=0}$$

$$f_A = \frac{1}{AE} \int_0^{3l} N_0 n_f ds + \frac{1}{I_x E} \int_0^{3l} M_0 m_f ds$$

• integrálok kiszámítása Simpson Formulával:

$$\int_0^{3l} N_0 n_f ds = \int_0^{2l} N_0 n_f ds + \int_{2l}^{3l} N_0 n_f ds = \frac{2l}{6} \left[F \frac{1}{l} + 4 F \frac{1}{l} + F \frac{1}{l} \right] = \frac{2l}{6} \cdot \frac{6F}{l} = 2F$$

$$\begin{aligned} \int_0^{3l} M_0 m_f ds &= \int_0^l M_0 m_f ds + \int_l^{2l} M_0 m_f ds + \int_{2l}^{3l} M_0 m_f ds = \\ &= \frac{l}{6} \left[0(-1) + 4 \cdot \left(-\frac{Fl}{2}\right)(-1) + (-Fl)(-1) \right] + \frac{l}{6} \left[(-Fl)(-1) + 4 \cdot (-Fl)(-1) + (-Fl)(-1) \right] + \frac{l}{6} \left[(-Fl)(-1) + 4 \cdot \left(-\frac{Fl}{2}\right)(-0.5) + 0 \cdot 0 \right] = \\ &= \frac{l}{6} \left[\frac{4Fl}{2} + Fl + 6Fl + Fl + Fl \right] = \frac{l}{6} \cdot 11Fl = \frac{11Fl^2}{6} \end{aligned}$$

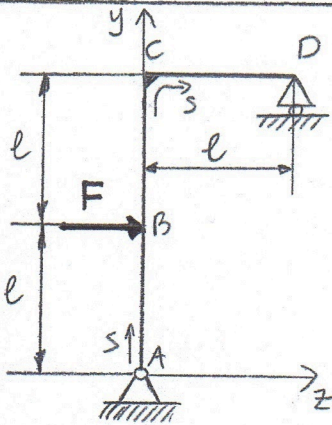
• behelyettesítve f_A képletébe

$$f_A = \frac{1}{AE} (2F) + \frac{1}{I_x E} \left(\frac{11Fl^2}{6} \right) = \frac{2F}{AE} + \frac{11Fl^2}{6 I_x E} =$$

$$= \frac{2 \cdot 3000}{1256,137 \cdot 2 \cdot 10^5} + \frac{11 \cdot 3000 \cdot 500^2}{6 \cdot 125663,7 \cdot 2 \cdot 10^5} = 0,055 \text{ rad}$$

$$0,055 \cdot \frac{180}{\pi} = \underline{\underline{3,15^\circ (\approx)}}$$

4.5 - TÖRTVONALÚ TARTÓ ALAKVÁLTOZÁSA



adott:

$$\begin{aligned} l &= 500 \text{ mm} \\ d &= 40 \text{ mm} \\ E &= 2 \cdot 10^5 \text{ MPa} \\ F &= 3000 \text{ N} \end{aligned}$$

Feladat:

B k.m. súlypontjának
z irányú elmozdulása
 $W_B = ?$

• az alakváltozási energia:

$$\begin{aligned} U &\approx U_N + U_H = \frac{1}{2} \int_0^{3l} \frac{N^2}{AE} ds + \frac{1}{2} \int_0^{3l} \frac{M_{Hx}^2}{Ix E} ds = \\ &= \frac{1}{2} \int_0^{3l} \frac{(F n_w)^2}{AE} ds + \frac{1}{2} \int_0^{3l} \frac{(F m_w)^2}{Ix E} ds \end{aligned}$$

• Castigliano-tétel:

$$W_B = \frac{\partial U}{\partial F} = \int_0^{3l} \frac{F n_w^2}{AE} ds + \int_0^{3l} \frac{F m_w^2}{Ix E} ds$$

$$W_B = \frac{F}{AE} \int_0^{3l} n_w^2 ds + \frac{F}{Ix E} \int_0^{3l} m_w^2 ds$$

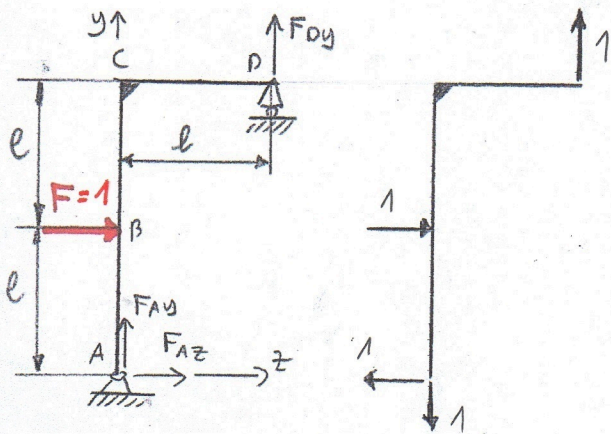
• integrálok kiszámítása Simpsonnal:

$$\int_0^{3l} n_w^2 ds = \frac{2l}{6} \left[1^2 + 4 \cdot 1^2 + 1^2 \right] = 2l$$

$$\begin{aligned} \int_0^{3l} m_w^2 ds &= \int_0^l m_w^2 ds + \int_l^{2l} m_w^2 ds + \int_{2l}^{3l} m_w^2 ds = \\ &= \frac{l}{6} \left[0^2 + 4 \left(-\frac{l}{2} \right)^2 + (-l)^2 \right] + \frac{l}{6} \left[-l^2 + 4(-l)^2 + (-l)^2 \right] + \\ &+ \frac{l}{6} \left[(-l)^2 + 4 \left(-\frac{l}{2} \right)^2 + 0^2 \right] = \frac{l}{6} [2l^2 + 6l^2 + 2l^2] = \\ &= \frac{l}{6} \cdot 10l^2 = \frac{5l^3}{3} \end{aligned}$$

• behelyettesítve W_B képletébe:

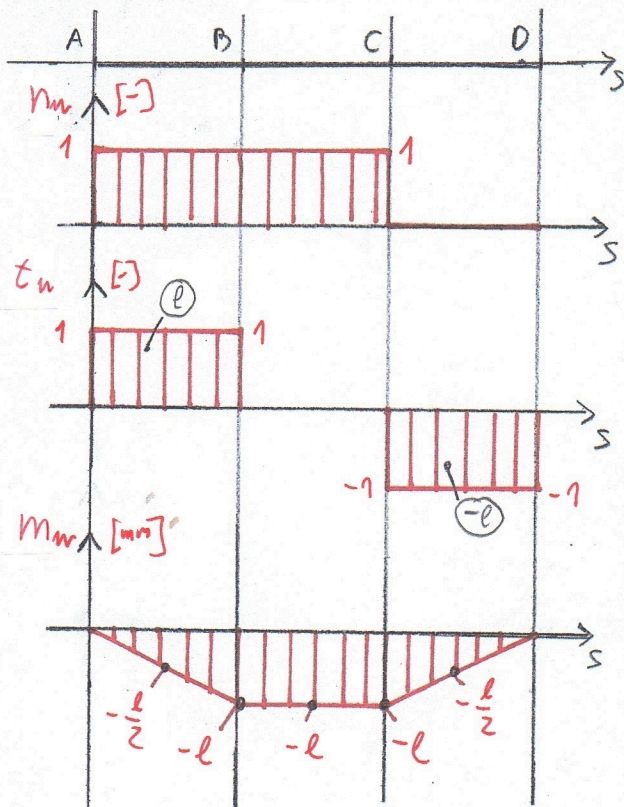
$$\begin{aligned} W_B &= \frac{F}{AE} (2l) + \frac{F}{Ix E} \left(\frac{5l^3}{3} \right) = \\ &= \frac{2lF}{AE} + \frac{5l^3 F}{3Ix E} = \\ &= \frac{2 \cdot 500 \cdot 3000}{1256,637 \cdot 2 \cdot 10^5} + \frac{5 \cdot 500^3 \cdot 3000}{3 \cdot 125663,7 \cdot 2 \cdot 10^5} = \\ &= \underline{\underline{24,88 \text{ mm} (-)}} \end{aligned}$$



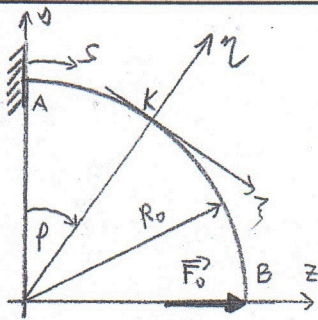
$$M_A = 0 = l - F_{By} l \Rightarrow F_{By} = 1$$

$$F_y = 0 = F_{Ay} + F_{By} \Rightarrow F_{Ay} = -F_{By} = -1$$

$$F_z = 0 = F_{Az} + 1 \Rightarrow F_{Az} = -1$$



4.6 - SÍKGÖRBE RÚD ALAKVÁLTOZÁSA



adott:

$$R_0 = 300 \text{ mm}$$

$$d = 80 \text{ mm}$$

$$\vec{F}_0 = (3\vec{e}_z) \text{ kN}$$

$$G = 0,8 \cdot 10^5 \text{ MPa}$$

$$\nu = 0,25$$

Feladat:

B keresztmetszet súlypontjának y irányú elmozdulása

$$V_B = ?$$

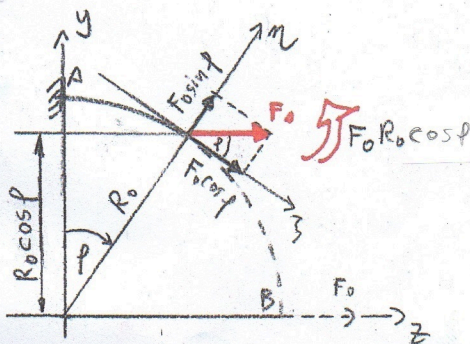
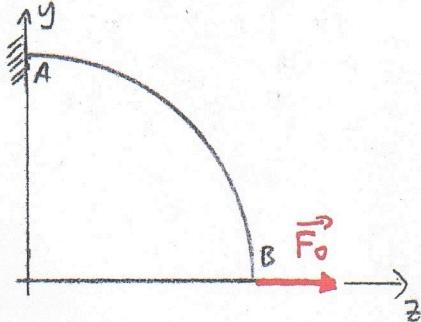
előszámítások:

$$E = 2G(1+\nu) = 2 \cdot 0,8 \cdot 10^5 (1+0,25) = 2 \cdot 10^5 \text{ MPa}$$

$$A = \frac{d^2 \pi}{4} = \frac{80^2 \pi}{4} = 5026,55 \text{ mm}^2$$

$$\frac{R_0}{d/2} = 7,5 > 3 \Rightarrow \ln x/x = \frac{d^4 \pi}{64} = \frac{80^4 \pi}{64} = 2010619,3 \text{ mm}^4$$

• eredeti terhelés



$$N_0 = F_0 \cos \varphi$$

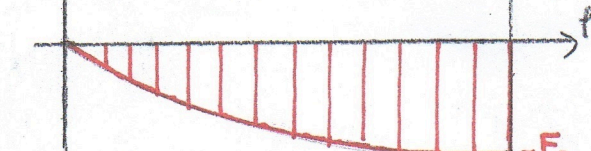
$$T_0 = -F_0 \sin \varphi$$

$$M_0 = -F_0 R_0 \cos \varphi$$

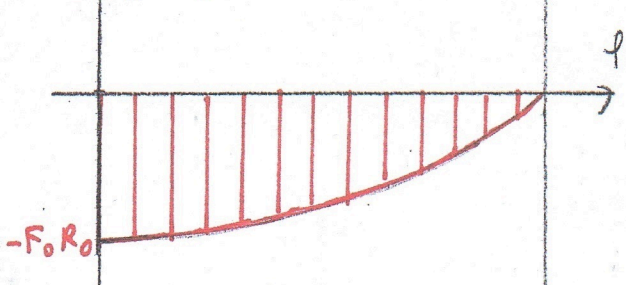
$$N_0 \text{ [N]}$$



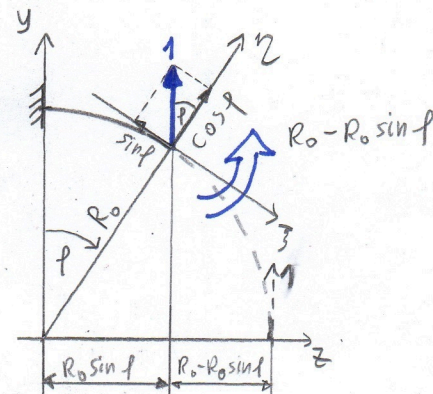
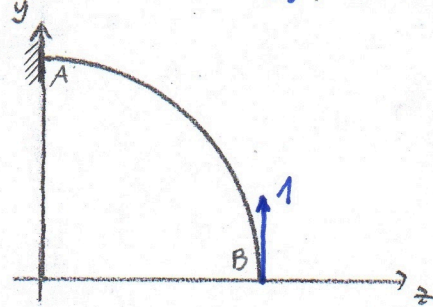
$$T_0 \text{ [N]}$$



$$M_0 \text{ [Nm]}$$



• egysegnyi F_0 terhelés

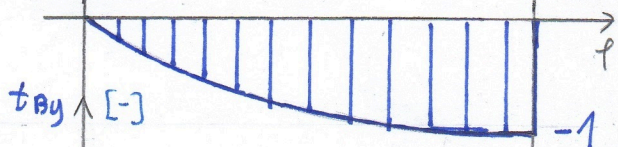


$$n_{By} = -\sin \varphi$$

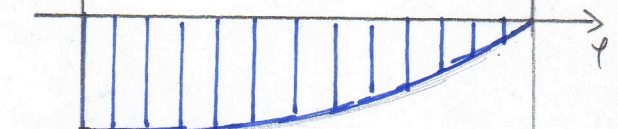
$$t_{By} = -\cos \varphi$$

$$m_{By} = -(R_0 - R_0 \sin \varphi) = -R_0 + R_0 \sin \varphi$$

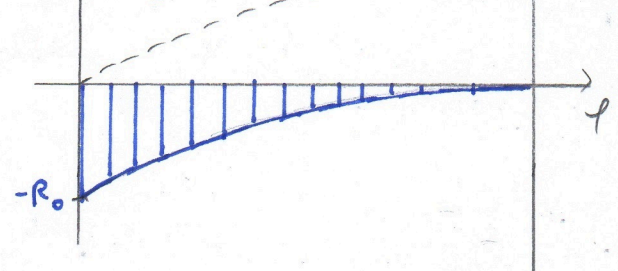
$$n_{By} \text{ [-]}$$



$$t_{By} \text{ [-]}$$



$$m_{By} \text{ [mm]}$$



• az alakváltozási energia:

$$U \approx U_N + U_H = \frac{1}{2} \int_{(e)} \frac{N^2}{AE} ds + \frac{1}{2} \int_{(e)} \frac{M_{hx}^2}{I \times E} ds = \frac{1}{2} \int_{l=0}^{\frac{\pi}{2}} \frac{(N_0 + F_{By} N_{By})^2}{AE} R_0 dl + \frac{1}{2} \int_{l=0}^{\frac{\pi}{2}} \frac{(M_0 + F_{By} M_{By})^2}{I \times E} R_0 dl$$

• Castigliano, tétel:

$$V_B = \left. \frac{\partial U}{\partial F_{By}} \right|_{F_{By}=0}$$

$$V_B = \int_{l=0}^{\frac{\pi}{2}} \frac{(N_0 + F_{By} N_{By}) N_{By}}{AE} R_0 dl + \int_{l=0}^{\frac{\pi}{2}} \frac{(M_0 + F_{By} M_{By}) M_{By}}{I \times E} R_0 dl \Big|_{F_{By}=0}$$

$$V_B = \frac{R_0}{AE} \int_{l=0}^{\frac{\pi}{2}} N_0 N_{By} dl + \frac{R_0}{I \times E} \int_{l=0}^{\frac{\pi}{2}} M_0 M_{By} dl$$

• integrálok kiszámítása:

$$\begin{aligned} \int_{l=0}^{\frac{\pi}{2}} N_0 N_{By} dl &= \int_{l=0}^{\frac{\pi}{2}} F_0 \cos l (-\sin l) dl = -F_0 \int_{l=0}^{\frac{\pi}{2}} \sin l \cos l dl = -\frac{F_0}{2} \int_{l=0}^{\frac{\pi}{2}} \sin 2l dl = \\ &= -\frac{F_0}{2} \left[-\frac{\cos 2l}{2} \right]_0^{\frac{\pi}{2}} = -\frac{F_0}{2} \left(-\frac{1}{2} - \frac{-1}{2} \right) = -\frac{F_0}{2} \end{aligned}$$

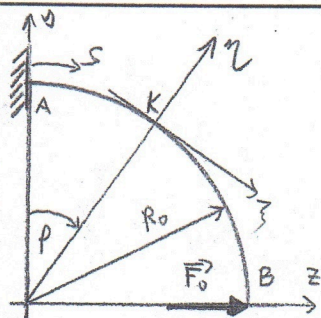
$$\begin{aligned} \int_{l=0}^{\frac{\pi}{2}} M_0 M_{By} dl &= \int_{l=0}^{\frac{\pi}{2}} -F_0 R_0 \cos l (-R_0 + R_0 \sin l) dl = \int_{l=0}^{\frac{\pi}{2}} F_0 R_0^2 \cos l dl + \int_{l=0}^{\frac{\pi}{2}} -F_0 R_0^2 \sin l \cos l dl = \\ &= F_0 R_0^2 \int_{l=0}^{\frac{\pi}{2}} \cos l dl - \frac{F_0 R_0^2}{2} \int_{l=0}^{\frac{\pi}{2}} \sin 2l dl = F_0 R_0^2 \left[\sin l \right]_0^{\frac{\pi}{2}} - \frac{F_0 R_0^2}{2} \left[-\frac{\cos 2l}{2} \right]_0^{\frac{\pi}{2}} = \\ &= F_0 R_0^2 (1-0) - \frac{F_0 R_0^2}{2} \left(-\frac{1}{2} - \frac{-1}{2} \right) = F_0 R_0^2 - \frac{F_0 R_0^2}{2} = \frac{F_0 R_0^2}{2} \end{aligned}$$

• behelyettesítve V_B képletébe

$$V_B = \frac{R_0}{AE} \left(-\frac{F_0}{2} \right) + \frac{R_0}{I \times E} \left(\frac{F_0 R_0^2}{2} \right) = -\frac{F_0 R_0}{AE} + \frac{F_0 R_0^3}{2 I \times E} =$$

$$= -\frac{3000 \cdot 300}{5026,55 \cdot 2 \cdot 10^5} + \frac{3000 \cdot 300^3}{2 \cdot 2010619,3 \cdot 2 \cdot 10^3} = \underline{\underline{0,1 \text{ mm} (\uparrow)}}$$

4.7 - SÍKGÖRBE RÚD ALAKVÁLTOZÁSA



adott:

$$R_0 = 300 \text{ mm}$$

$$d = 80 \text{ mm}$$

$$\vec{F}_0 = (3\vec{e}_z) \text{ kN}$$

$$G = 0,8 \cdot 10^5 \text{ MPa}$$

$$\nu = 0,25$$

Feladat:

B keresztmetszet súlypontjának z irányú elmozdulása

$$w_B = ?$$

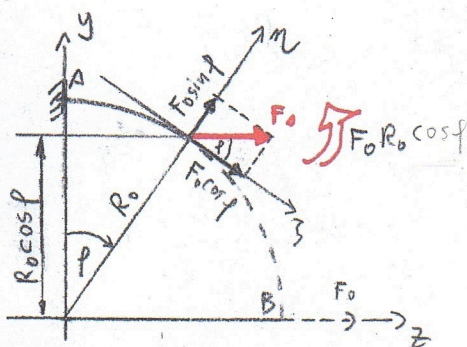
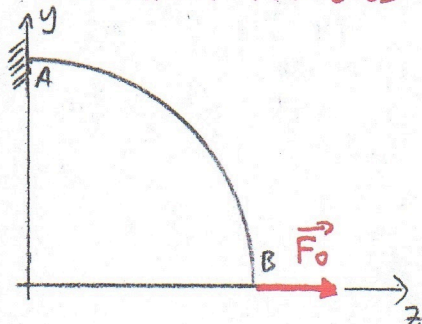
előszámítások:

$$E = 2G(1+\nu) = 2 \cdot 0,8 \cdot 10^5 (1+0,25) = 2 \cdot 10^5 \text{ MPa}$$

$$A = \frac{d^2 \pi}{4} = \frac{80^2 \pi}{4} = 5026,55 \text{ mm}^2$$

$$\frac{R_0}{d/2} = 7,5 > 3 \Rightarrow \ln \pi x = \frac{d^4 \pi}{64} = \frac{80^4 \pi}{64} = 2010619,3 \text{ mm}^4$$

• eredeti terhelés



$$N_0 = F_0 \cos \phi$$

$$T_0 = -F_0 \sin \phi$$

$$M_0 = -F_0 R_0 \cos \phi$$

$$N_0 \text{ [N]}$$

$$F_0$$

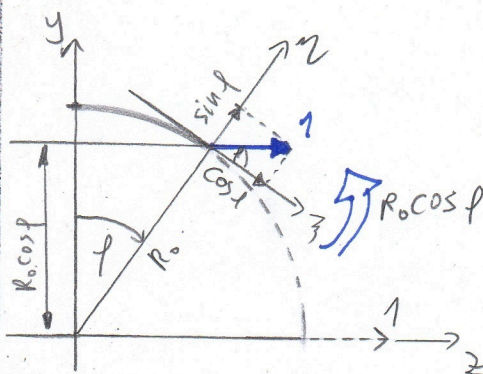
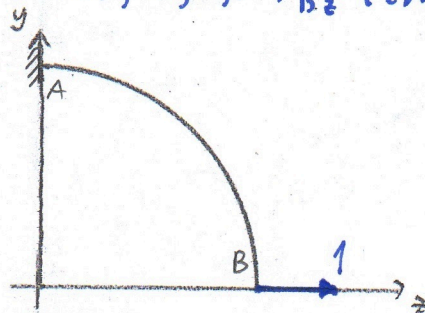
$$T_0 \text{ [N]}$$

$$-F_0$$

$$M_0 \text{ [Nm]}$$

$$-F_0 R_0$$

• egységnyi F_{Bz} terhelés



$$n_{Bz} = \cos \phi$$

$$t_{Bz} = -\sin \phi$$

$$m_{Bz} = -R_0 \cos \phi$$

$$n_{Bz} \text{ [-]}$$

$$1$$

$$t_{Bz} \text{ [-]}$$

$$-1$$

$$m_{Bz} \text{ [mm]}$$

$$-R_0$$

• az alakváltozási energia:

$$U \approx U_N + U_H = \frac{1}{2} \int_{(l)} \frac{N^2}{AE} ds + \frac{1}{2} \int_{(l)} \frac{M_{Hx}^2}{I_x E} ds = \frac{1}{2} \int_{\varphi=0}^{\frac{\pi}{2}} \frac{(N_0 + F_{Bz} N_{Bz})^2}{AE} R_0 d\varphi + \frac{1}{2} \int_{\varphi=0}^{\frac{\pi}{2}} \frac{(M_0 + F_{Bz} M_{Bz})^2}{I_x E} R_0 d\varphi$$

• Castigliano tétel:

$$W_B = \left. \frac{\partial U}{\partial F_{Bz}} \right|_{F_{Bz}=0}$$

$$W_B = \int_{\varphi=0}^{\frac{\pi}{2}} \frac{(N_0 + F_{Bz} N_{Bz}) N_{Bz}}{AE} R_0 d\varphi \Big|_{F_{Bz}=0} + \int_{\varphi=0}^{\frac{\pi}{2}} \frac{(M_0 + F_{Bz} M_{Bz}) M_{Bz}}{I_x E} R_0 d\varphi \Big|_{F_{Bz}=0}$$

$$W_B = \frac{R_0}{AE} \int_{\varphi=0}^{\frac{\pi}{2}} N_0 N_{Bz} d\varphi + \frac{R_0}{I_x E} \int_{\varphi=0}^{\frac{\pi}{2}} M_0 M_{Bz} d\varphi$$

• integrálok kiszámítása

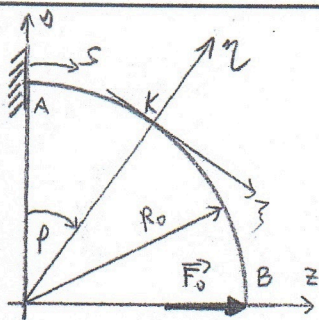
$$\begin{aligned} \int_{\varphi=0}^{\frac{\pi}{2}} N_0 N_{Bz} d\varphi &= \int_{\varphi=0}^{\frac{\pi}{2}} F_0 \cos \varphi \cos \varphi d\varphi = F_0 \int_{\varphi=0}^{\frac{\pi}{2}} \cos^2 \varphi d\varphi = \frac{F_0}{2} \int_{\varphi=0}^{\frac{\pi}{2}} (1 + \cos 2\varphi) d\varphi = \frac{F_0}{2} \left[\varphi + \frac{\sin 2\varphi}{2} \right]_0^{\frac{\pi}{2}} = \\ &= \frac{F_0}{2} \left(\left(\frac{\pi}{2} + 0 \right) - (0 + 0) \right) = \frac{F_0 \pi}{4} \end{aligned}$$

$$\begin{aligned} \int_{\varphi=0}^{\frac{\pi}{2}} M_0 M_{Bz} d\varphi &= \int_{\varphi=0}^{\frac{\pi}{2}} -F_0 R_0 \cos \varphi \cdot (-R_0 \cos \varphi) d\varphi = \int_{\varphi=0}^{\frac{\pi}{2}} F_0 R_0^2 \cos^2 \varphi d\varphi = \frac{F_0 R_0^2}{2} \int_{\varphi=0}^{\frac{\pi}{2}} (1 + \cos 2\varphi) d\varphi = \\ &= \frac{F_0 R_0^2}{2} \left[\varphi + \frac{\sin 2\varphi}{2} \right]_0^{\frac{\pi}{2}} = \frac{F_0 R_0^2}{2} \left(\left(\frac{\pi}{2} + 0 \right) - (0 + 0) \right) = \frac{F_0 R_0^2 \pi}{4} \end{aligned}$$

• behelyettesítve W_B képletébe

$$\begin{aligned} W_B &= \frac{R_0}{AE} \left(\frac{F_0 \pi}{4} \right) + \frac{R_0}{I_x E} \left(\frac{F_0 R_0^2 \pi}{4} \right) = \frac{F_0 R_0 \pi}{4AE} + \frac{F_0 R_0^3 \pi}{4I_x E} = \\ &= \frac{3000 \cdot 300 \pi}{4 \cdot 5026,55 \cdot 2 \cdot 10^5} + \frac{3000 \cdot 300^3 \pi}{4 \cdot 2010619,3 \cdot 2 \cdot 10^5} = \underline{\underline{0,159 \text{ mm} (\rightarrow)}} \end{aligned}$$

4.8 - SÍKGÖRBE RÚD ALAKVÁLTOZÁSA



adott:

$$R_0 = 300 \text{ mm}$$

$$d = 80 \text{ mm}$$

$$\vec{F}_0 = (3\vec{e}_z) \text{ kN}$$

$$G = 0,8 \cdot 10^5 \text{ MPa}$$

$$\nu = 0,25$$

Feladat:

B kővetési metszet súlypontjának z irányú elmozdulása

$$w_B = ?$$

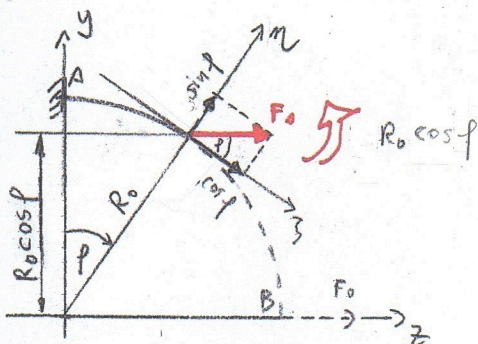
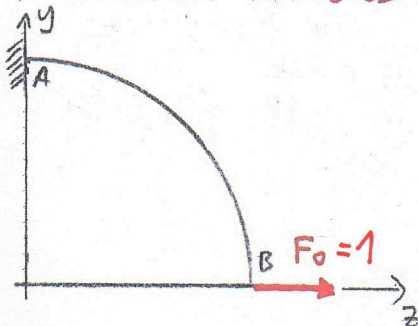
előszámitások:

$$E = 2G(1+\nu) = 2 \cdot 0,8 \cdot 10^5 (1+0,25) = 2 \cdot 10^5 \text{ MPa}$$

$$A = \frac{d^2 \pi}{4} = \frac{80^2 \pi}{4} = 5026,55 \text{ mm}^2$$

$$\frac{R_0}{d/2} = 7,5 > 3 \Rightarrow \ln x = \frac{d^4 \pi}{64} = \frac{80^4 \pi}{64} = 2010619,3 \text{ mm}^4$$

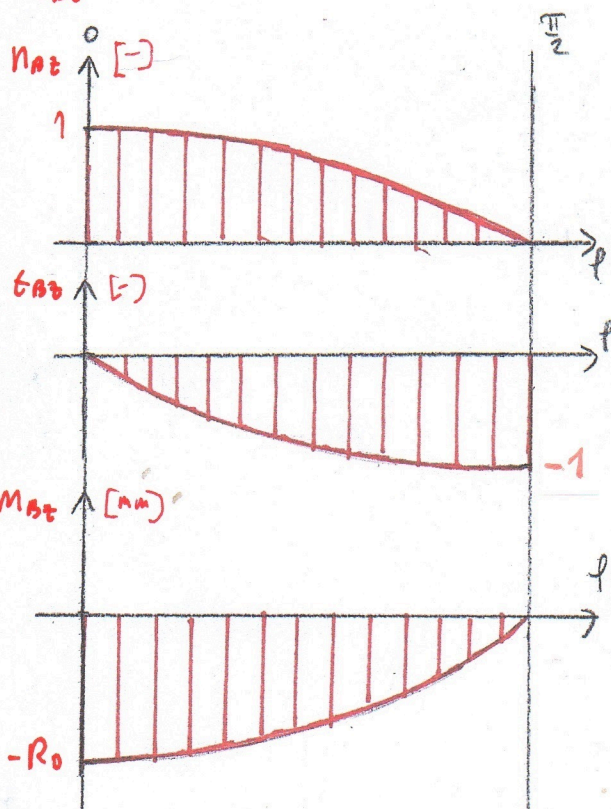
• eredeti terhelés



$$n_{Bz} = \cos \varphi$$

$$t_{Bz} = -\sin \varphi$$

$$m_{Bz} = -R_0 \cos \varphi$$



• az alakváltozási energia:

$$U \propto U_N + U_H = \frac{1}{2} \int_{(e)} \frac{N^2}{AE} ds + \frac{1}{2} \int_{(e)} \frac{M_{Hx}^2}{I_x E} ds =$$

$$= \frac{1}{2} \int_{\varphi=0}^{\frac{\pi}{2}} \frac{(F_0 n_{Bz})^2}{AE} R_0 d\varphi + \frac{1}{2} \int_{\varphi=0}^{\frac{\pi}{2}} \frac{(F_0 m_{Bz})^2}{I_x E} R_0 d\varphi$$

• Castigliano-tétel:

$$w_B = \frac{\partial U}{\partial F_0} = \int_{\varphi=0}^{\frac{\pi}{2}} \frac{F_0 n_{Bz}^2}{AE} R_0 d\varphi + \int_{\varphi=0}^{\frac{\pi}{2}} \frac{F_0 m_{Bz}^2}{I_x E} R_0 d\varphi =$$

$$\frac{F_0 R_0}{AE} \int_{\varphi=0}^{\frac{\pi}{2}} n_{Bz}^2 d\varphi + \frac{F_0 R_0}{I_x E} \int_{\varphi=0}^{\frac{\pi}{2}} m_{Bz}^2 d\varphi =$$

• integrálók leírásai:

$$\int_{\varphi=0}^{\frac{\pi}{2}} n_{Bz}^2 d\varphi = \int_{\varphi=0}^{\frac{\pi}{2}} \cos^2 \varphi d\varphi = \frac{1}{2} \int_{\varphi=0}^{\frac{\pi}{2}} 1 + \cos 2\varphi d\varphi =$$

$$= \frac{1}{2} \left[\varphi + \frac{\sin 2\varphi}{2} \right]_0^{\frac{\pi}{2}} = \frac{1}{2} \frac{\pi}{2} = \frac{\pi}{4}$$

$$\int_{\varphi=0}^{\frac{\pi}{2}} m_{Bz}^2 d\varphi = \int_{\varphi=0}^{\frac{\pi}{2}} (-R_0 \cos \varphi)^2 d\varphi = R_0^2 \int_{\varphi=0}^{\frac{\pi}{2}} \cos^2 \varphi d\varphi =$$

$$= \frac{R_0^2}{2} \int_{\varphi=0}^{\frac{\pi}{2}} 1 + \cos 2\varphi d\varphi = \frac{R_0^2}{2} \left[\varphi + \frac{\sin 2\varphi}{2} \right]_0^{\frac{\pi}{2}} = \frac{R_0^2 \pi}{4}$$

• behelyettesítve w_B képletébe

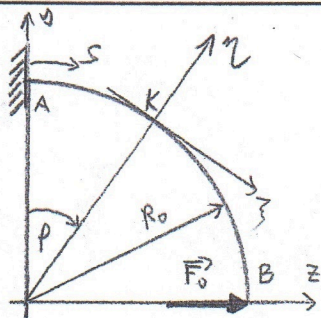
$$w_B = \frac{F_0 R_0}{AE} \left(\frac{\pi}{4} \right) + \frac{F_0 R_0}{I_x E} \left(\frac{R_0^2 \pi}{4} \right) =$$

$$= \frac{F_0 R_0 \pi}{4AE} + \frac{F_0 R_0^3 \pi}{4I_x E} =$$

$$= \frac{3000 \cdot 300 \cdot \pi}{4 \cdot 5026,53 \cdot 2 \cdot 10^5} + \frac{3000 \cdot 300^3 \cdot \pi}{4 \cdot 2010619,3 \cdot 2 \cdot 10^5} =$$

$$= 0,159 \text{ mm} \rightarrow$$

4.9 - SÍKGÖRBE RÚD ALAKVÁLTOZÁSA



adott:

$$R_0 = 300 \text{ mm}$$

$$d = 80 \text{ mm}$$

$$\vec{F}_0 = (3\vec{e}_z) \text{ kN}$$

$$G = 0,8 \cdot 10^5 \text{ MPa}$$

$$\nu = 0,25$$

Feladat:

B keresztmetszet x tengely körüli szögfordulása

$$\varphi_{Bx} = ?$$

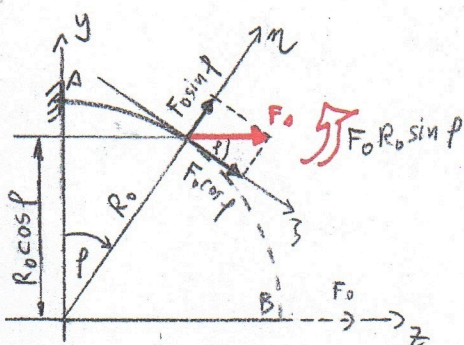
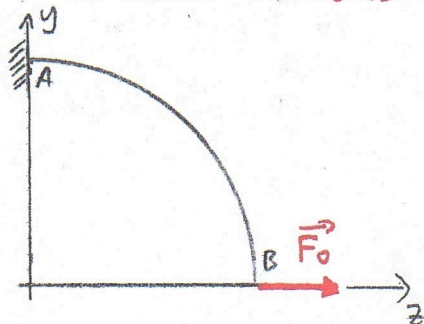
előszámítások:

$$E = 2G(1+\nu) = 2 \cdot 0,8 \cdot 10^5 (1+0,25) = 2 \cdot 10^5 \text{ MPa}$$

$$A = \frac{d^2 \pi}{4} = \frac{80^2 \pi}{4} = 5026,55 \text{ mm}^2$$

$$\frac{R_0}{d/2} = 7,5 > 3 \Rightarrow \ln \frac{R_0}{d/2} = \frac{d^4 \pi}{64} = \frac{80^4 \pi}{64} = 2010619,3 \text{ mm}^4$$

• eredeti terhelés



$$N_0 = F_0 \cos \varphi$$

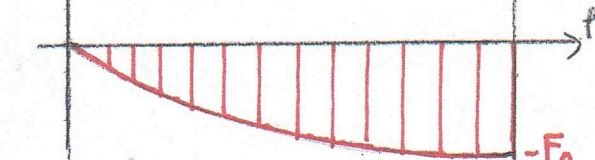
$$T_0 = -F_0 \sin \varphi$$

$$M_0 = -F_0 R_0 \cos \varphi$$

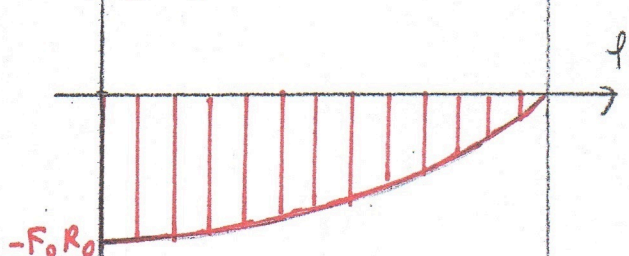
$$N_0 \text{ [N]}$$



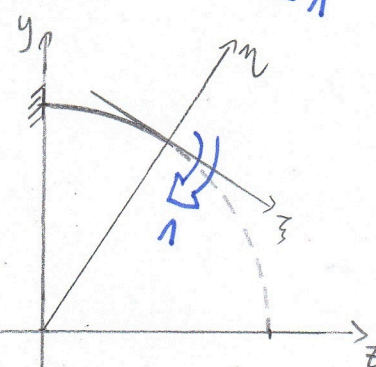
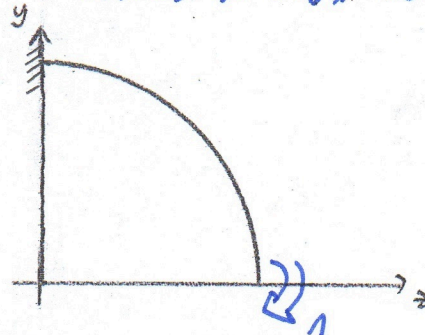
$$T_0 \text{ [N]}$$



$$M_0 \text{ [Nm]}$$



• egysegnyi M_{\theta x} terhelés

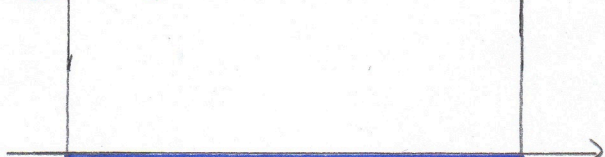


$$N_{Bx} = 0$$

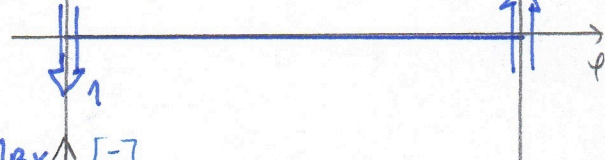
$$t_{\theta x} = 0$$

$$M_{Bx} = 1$$

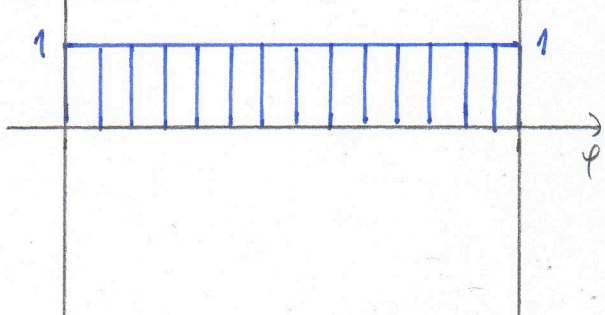
$$N_{\theta x} \text{ [N]}$$



$$t_{\theta x} \text{ [N/mm]}$$



$$M_{\theta x} \text{ [-]}$$



• az alakváltozási energia:

$$U \approx U_N + U_H = \frac{1}{2} \int_{(e)} \frac{N^2}{AE} ds + \frac{1}{2} \int_{(e)} \frac{M_{hx}^2}{I_x E} ds = \frac{1}{2} \int_{\varphi=0}^{\frac{\pi}{2}} \frac{(N_0 + M_{0x} n_{Bx})^2}{AE} R_0 d\varphi + \frac{1}{2} \int_{\varphi=0}^{\frac{\pi}{2}} \frac{(M_0 + M_{0x} m_{Bx})^2}{I_x E} R_0 d\varphi$$

• Castigliano-tétel:

$$\varphi_{Bx} = \left. \frac{\partial U}{\partial M_{Bx}} \right|_{M_{Bx}=0}$$

$$\varphi_{Bx} = \int_{\varphi=0}^{\frac{\pi}{2}} \frac{(N_0 + M_{0x} n_{Bx}) n_{Bx}}{AE} R_0 d\varphi \Big|_{M_{0x}=0} + \int_{\varphi=0}^{\frac{\pi}{2}} \frac{(M_0 + M_{0x} m_{Bx}) m_{Bx}}{I_x E} R_0 d\varphi \Big|_{M_{0x}=0}$$

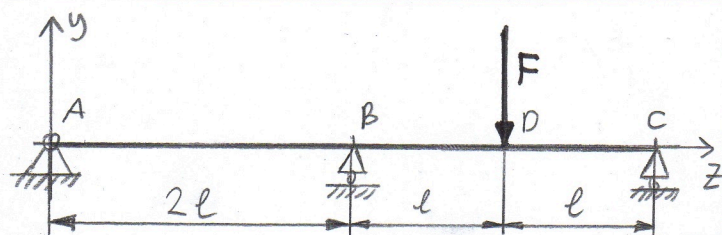
$$\varphi_{Bx} = \frac{R_0}{AE} \int_{\varphi=0}^{\frac{\pi}{2}} \underbrace{N_0}_{=0} n_{Bx} d\varphi + \frac{R_0}{I_x E} \int_{\varphi=0}^{\frac{\pi}{2}} M_0 m_{Bx} d\varphi = \frac{R_0}{I_x E} \int_{\varphi=0}^{\frac{\pi}{2}} M_0 m_{Bx} d\varphi =$$

$$= \frac{R_0}{I_x E} \int_{\varphi=0}^{\frac{\pi}{2}} -F_0 R_0 \cos \varphi \cdot 1 d\varphi = - \frac{F_0 R_0^2}{I_x E} \int_{\varphi=0}^{\frac{\pi}{2}} \cos \varphi d\varphi = - \frac{F_0 R_0^2}{I_x E} [\sin \varphi]_0^{\frac{\pi}{2}} =$$

$$= - \frac{F_0 R_0^2}{I_x E} (1-0) = - \frac{F_0 R_0^2}{I_x E} = - \frac{3000 \cdot 300^2}{2010613,3 \cdot 2 \cdot 10^5} = -6,71 \cdot 10^{-4} \text{ rad}$$

$$-6,71 \cdot 10^{-4} \cdot \frac{180}{\pi} = \underline{\underline{-0,0384^\circ (^\circ)}}$$

4.10 - STATIKAILAG HATÁROZATLAN EGYENES RÚD

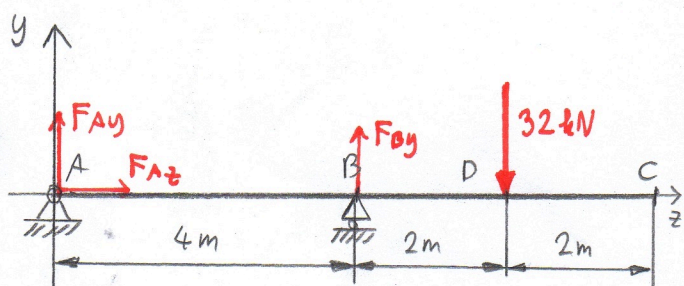


adott:
 $l = 2\text{ m}$
 $F = 32\text{ kN}$
 $I_x = 50000\text{ mm}^4$
 $E = 2,1 \cdot 10^5\text{ MPa}$

Feladat:
 • támasztóerők
 • igénybevételi ábrák

ismeretlen támasztóerők: $F_{Ay}, F_{Az}, F_{By}, F_{Cy} \Rightarrow 4\text{ db}$
 statikai egyenletek: $F_y = 0; F_z = 0; M_a = 0 \Rightarrow 3\text{ db}$
 $\Rightarrow$ A tartó statikailag egyszerűen határozatlan
 A határozottá tétel egy lehetséges módja: C görögö elhagyása

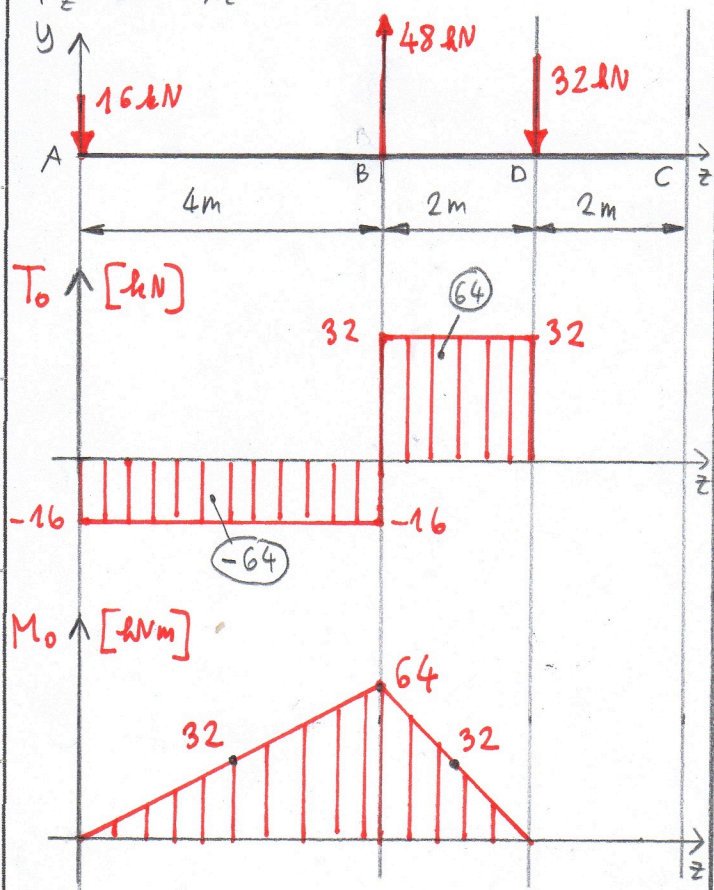
eredeti terhelés a határozott ártett tartón



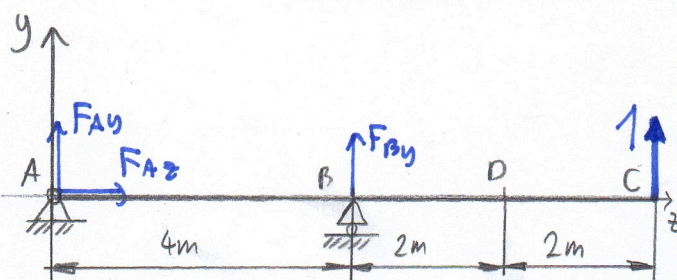
$$M_a = 0 = 6 \cdot 32 - 4 F_{By} \Rightarrow F_{By} = 48\text{ kN} (\uparrow)$$

$$M_b = 0 = 2 \cdot 32 + 4 F_{Ay} \Rightarrow F_{Ay} = -16\text{ kN} (\downarrow)$$

$$F_z = 0 = F_{Az}$$



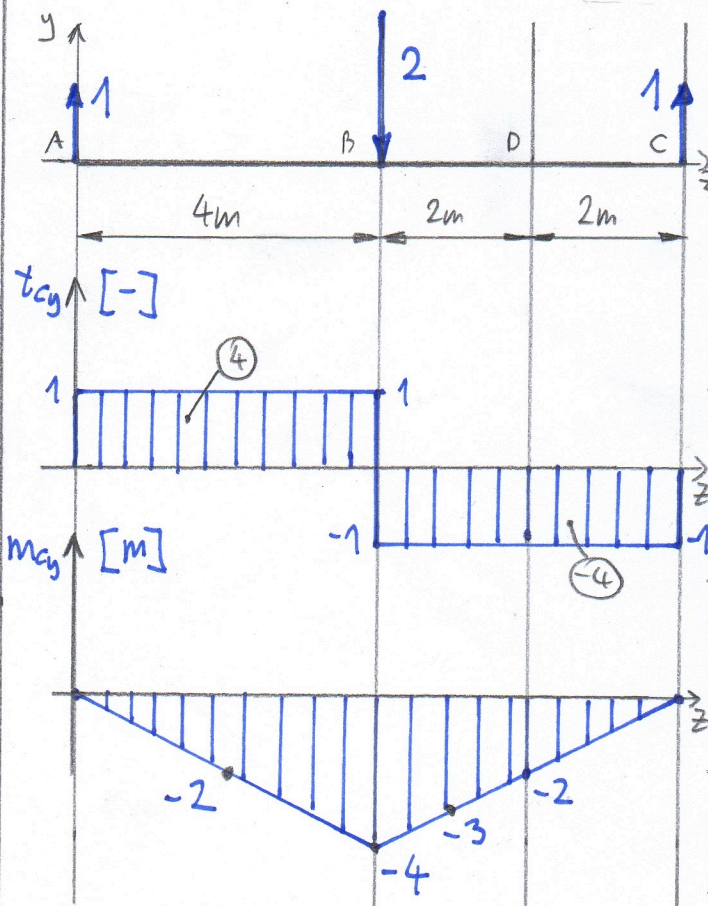
olt elhagyott támasztóerővelc megfelelő egysegnyi nagyságú terhelés



$$M_a = 0 = -8 \cdot 1 - 4 F_{By} \Rightarrow F_{By} = -2 (\downarrow)$$

$$M_b = 0 = -4 \cdot 1 + 4 F_{Ay} \Rightarrow F_{Ay} = 1 (\uparrow)$$

$$F_z = 0 = F_{Az}$$



• az alakváltozási energia:

$$U \approx U_H = \frac{1}{2} \int_0^{4l} \frac{M_{hx}^2}{I_x E} dz = \frac{1}{2} \int_0^{4l} \frac{(M_0 + F_{cy} m_{cy})^2}{I_x E} dz$$

• Castigliano -tétel + kinematikai peremfeltétel:

$$V_c = 0 = \frac{\partial U}{\partial F_{cy}} = \int_0^{4l} \frac{(M_0 + F_{cy} m_{cy}) m_{cy}}{I_x E} dz \cdot I_x E$$

$$0 = \int_0^{4l} M_0 m_{cy} + F_{cy} m_{cy}^2 dz$$

$$0 = \int_0^{4l} M_0 m_{cy} dz + F_{cy} \int_0^{4l} m_{cy}^2 dz$$

$$F_{cy} = - \frac{\int_0^{4l} M_0 m_{cy} dz}{\int_0^{4l} m_{cy}^2 dz}$$

• integrálok kiszámítása Simpson formulával:

$$\int_0^{4l} M_0 m_{cy} dz = \int_0^{2l} M_0 m_{cy} dz + \int_{2l}^{3l} M_0 m_{cy} dz + \underbrace{\int_{3l}^{4l} M_0 m_{cy} dz}_{=0} =$$

$$= \frac{4}{6} [0 \cdot 0 + 4 \cdot 32 \cdot (-2) + 64 \cdot (-4)] + \frac{2}{6} [64 \cdot (-4) + 4 \cdot 32 \cdot (-3) + 0 \cdot (-2)] =$$

$$= \frac{4}{6} [-512] + \frac{2}{6} [-640] = -\frac{2048}{6} - \frac{1280}{6} = -\frac{3328}{6} \text{ kNm}^3$$

$$\int_0^{4l} m_{cy}^2 dz = \int_0^{2l} m_{cy}^2 dz + \int_{2l}^{4l} m_{cy}^2 dz =$$

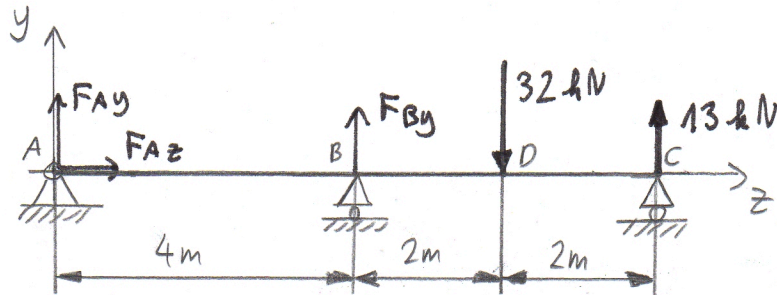
$$= \frac{4}{6} [0^2 + 4 \cdot (-2)^2 + (-4)^2] + \frac{4}{6} [(-4)^2 + 4 \cdot (-2)^2 + 0^2] =$$

$$= \frac{4}{6} [32] + \frac{4}{6} [32] = \frac{128}{6} + \frac{128}{6} = \frac{256}{6} \text{ m}^3$$

• visszahelyettesítve F_{cy} képletébe:

$$F_{cy} = - \frac{-\frac{3328}{6}}{\frac{296}{6}} = - \frac{-3328}{296} = 11.24 \text{ kN} (\uparrow)$$

• többi támasztóerő $\rightarrow$ statikai egyenletekből:



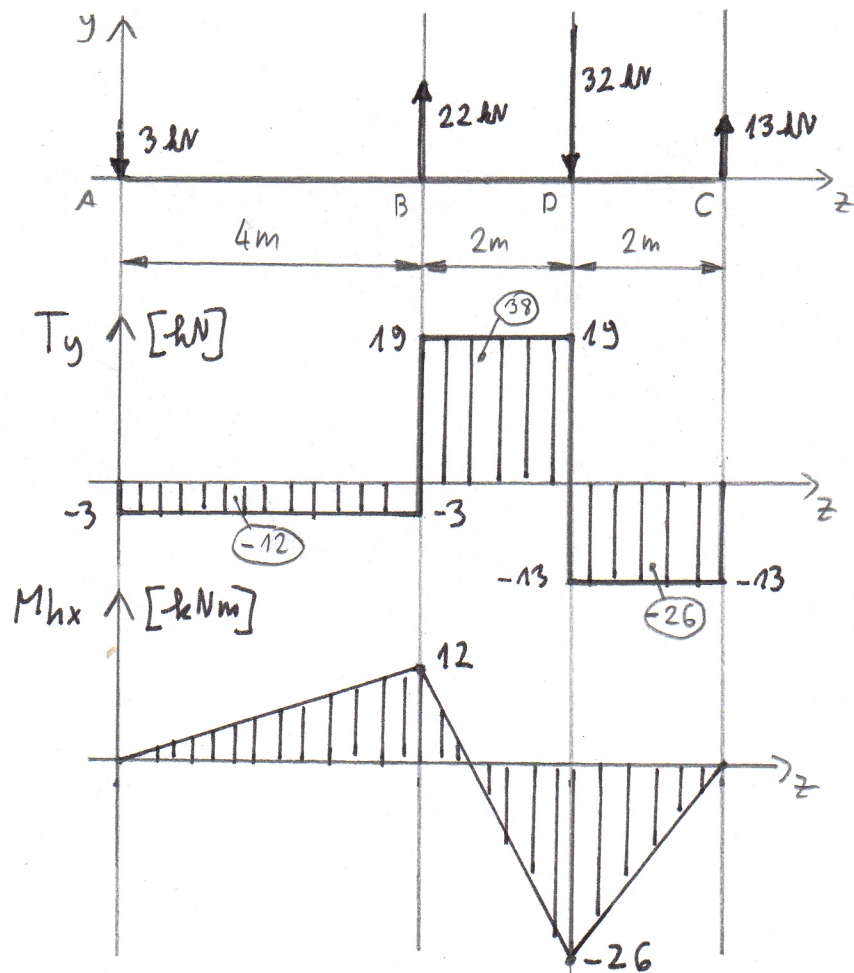
$$M_a = 0 = -4F_{By} + 6 \cdot 32 - 8 \cdot 13 \Rightarrow F_{By} = \frac{8 \cdot 13 - 6 \cdot 32}{-4} = 22 \text{ kN} (\uparrow)$$

$$M_b = 0 = 4F_{Ay} + 2 \cdot 32 - 4 \cdot 13 \Rightarrow F_{Ay} = \frac{4 \cdot 13 - 2 \cdot 32}{4} = -3 \text{ kN} (\downarrow)$$

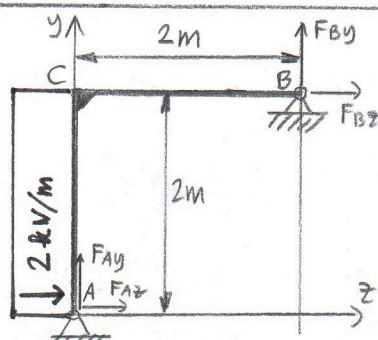
$$F_z = 0 = F_{Az}$$

$$\vec{F}_A = (-3\vec{e}_y) \text{ kN}; \quad \vec{F}_B = (22\vec{e}_y) \text{ kN}; \quad \vec{F}_C = (13\vec{e}_y) \text{ kN}$$

• eredeti szerkezet igénybevételei ábrái:



4.11 - STATIKAILAG HATÁROZATLAN TÖRTVONALÚ TARTÓ

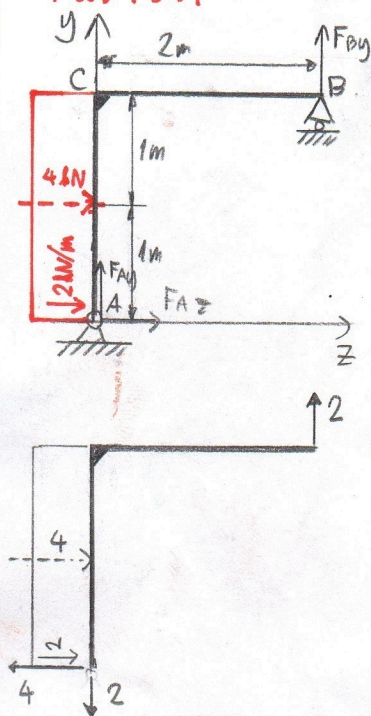


- ismeretlen támasztóerők: $F_{Ay}, F_{Az}, F_{By}, F_{Bz} \Rightarrow 4 \text{ db}$
- statikai egyenletek: $F_y = 0, F_z = 0, M_a = 0 \Rightarrow 3 \text{ db}$
- A tartó statikailag egyszerűen határozatlan

$$d = 100 \text{ mm} \Rightarrow A = \frac{d^2 \pi}{4} = \frac{0,1^2 \pi}{4} = 7,85398 \cdot 10^{-3} \text{ m}^2$$

$$I_x = \frac{d^4 \pi}{64} = \frac{0,1^4 \pi}{64} = 4,90873 \cdot 10^{-6} \text{ m}^4$$

• eredeti terhelés a határozottá tett tartón



- támasztóerők:

$$M_a = 0 = 1 \cdot 4 - 2 F_{By}$$

$\Downarrow$

$$F_{By} = \frac{4}{2} = 2 \text{ kN} (\uparrow)$$

$$F_y = 0 = F_{Ay} + F_{By}$$

$\Downarrow$

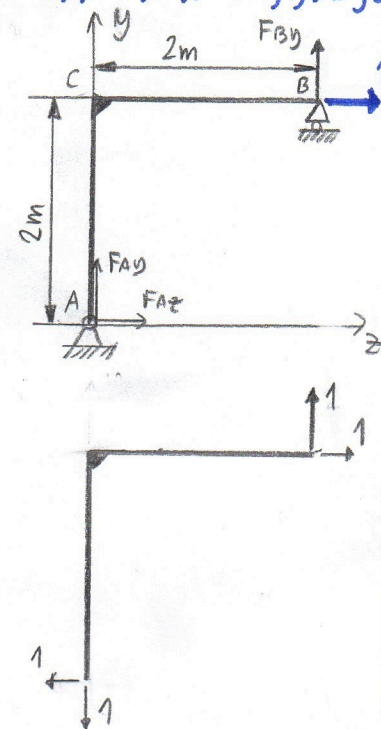
$$F_{Ay} = -F_{By} = -2 \text{ kN} (\downarrow)$$

$$F_z = 0 = F_{Az} + 4$$

$\Downarrow$

$$F_{Az} = -4 \text{ kN} (\leftarrow)$$

• az elhagyott támasztóerőnek megfelelő egységnyi nagyságú terhelés



- támasztóerők

$$M_a = 0 = 2 \cdot 1 - 2 F_{By}$$

$\Downarrow$

$$F_{By} = \frac{2 \cdot 1}{2} = 1 (\uparrow)$$

$$F_y = 0 = F_{Ay} + F_{By}$$

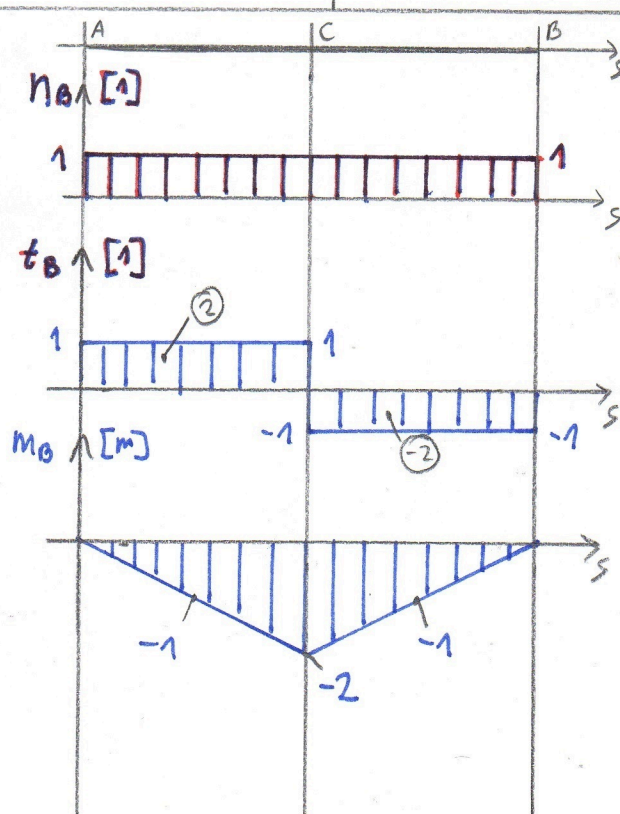
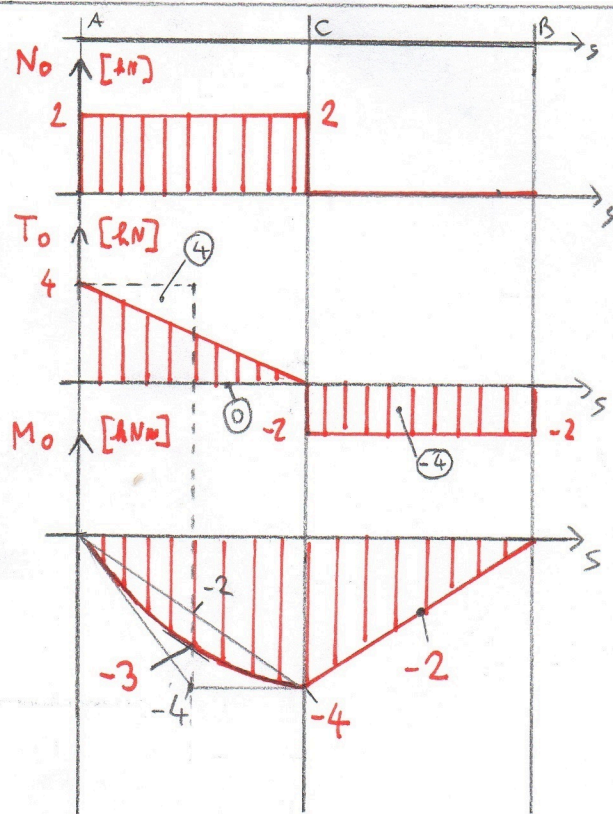
$\Downarrow$

$$F_{Ay} = -F_{By} = -1 (\downarrow)$$

$$F_z = 0 = F_{Az} + 1$$

$\Downarrow$

$$F_{Az} = -1 (\leftarrow)$$



• az alakváltozási energia:

$$U \approx U_N + U_H = \frac{1}{2} \int_A^B \frac{N^2}{AE} ds + \frac{1}{2} \int_A^B \frac{M^2}{I_x E} ds = \frac{1}{2} \int_A^B \frac{(N_0 + F_{Bz} n_B)^2}{AE} ds + \frac{1}{2} \int_A^B \frac{(M_0 + F_{Bz} m_B)^2}{I_x E} ds$$

• Castigliano - tétel:

$$w_B = 0 = \frac{\partial U}{\partial F_{Bz}} = \int_A^B \frac{(N_0 + F_{Bz} n_B) n_B}{AE} ds + \int_A^B \frac{(M_0 + F_{Bz} m_B) m_B}{I_x E} ds \quad / \cdot E$$

$$0 = \frac{1}{A} \int_A^B N_0 n_B ds + F_{Bz} \frac{1}{A} \int_A^B n_B^2 ds + \frac{1}{I_x} \int_A^B M_0 m_B ds + F_{Bz} \frac{1}{I_x} \int_A^B m_B^2 ds$$

$$F_{Bz} = \frac{-\frac{1}{A} \int_A^B N_0 n_B ds - \frac{1}{I_x} \int_A^B M_0 m_B ds}{\frac{1}{A} \int_A^B n_B^2 ds + \frac{1}{I_x} \int_A^B m_B^2 ds}$$

• integrálok kiszámítása:

$$\int_A^B N_0 n_B ds = \int_A^C N_0 n_B ds = \frac{2}{6} \left[2 \cdot 1 + 4 \cdot 2 \cdot 1 + 2 \cdot 1 \right] = 4$$

$$\int_A^B M_0 m_B ds = \int_A^C M_0 m_B ds + \int_C^B M_0 m_B ds = \frac{2}{6} \left[0 \cdot 0 + 4 \cdot (-3) \cdot (-1) + (-4) \cdot (-2) \right] + \frac{2}{6} \left[-4 \cdot (-2) + 4 \cdot (-2) \cdot (-1) + 0 \cdot 0 \right] =$$

$$= \frac{2}{6} [20 + 16] = 12$$

$$\int_A^B n_B^2 ds = \frac{4}{6} [1^2 + 4 \cdot 1^2 + 1^2] = 4$$

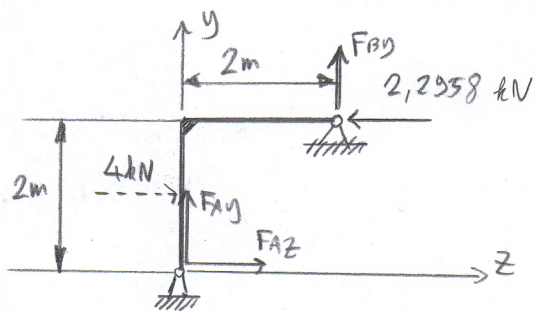
$$\int_A^B m_B^2 ds = \int_A^C m_B^2 ds + \int_C^B m_B^2 ds = \frac{2}{6} [0^2 + 4 \cdot (-1)^2 + (-2)^2] + \frac{2}{6} [(-2)^2 + 4 \cdot (-1)^2 + 0^2] =$$

$$= \frac{2}{6} [8 + 8] = \frac{16}{3}$$

• behelyettesítve F_{Bz} képletébe:

$$F_{Bz} = \frac{-\frac{1}{A} \cdot 4 - \frac{1}{I_x} \cdot 12}{\frac{1}{A} \cdot 4 + \frac{1}{I_x} \cdot \frac{16}{3}} = \frac{-\frac{1}{7,85339 \cdot 10^{-3}} - \frac{3}{4,90873 \cdot 10^{-6}}}{\frac{1}{7,85339 \cdot 10^{-3}} + \frac{4}{3 \cdot 4,90873 \cdot 10^{-6}}} = -2,2958 \text{ kN} (\leftarrow)$$

többi támasztóerő $\Rightarrow$ statikai egyenletekből



$$F_z = 0 = F_{Az} + 4 - 2,2958 \Rightarrow F_{Az} = -1,7042 \text{ kN} (\leftarrow)$$

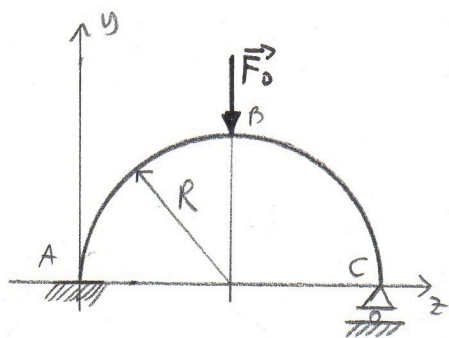
$$M_a = 0 = 1 \cdot 4 - 2 F_{By} - 2 \cdot 2,2958 \Rightarrow F_{By} = \frac{2 \cdot 2,2958 - 1 \cdot 4}{-2} = -0,2958 \text{ kN} (\downarrow)$$

$$F_y = 0 = F_{Ay} - 0,2958 \Rightarrow F_{Ay} = 0,2958 \text{ kN} (\uparrow)$$

$$\vec{F}_A = (0,2958 \vec{e}_y - 1,7042 \vec{e}_z) \text{ kN}$$

$$\vec{F}_B = (-0,2958 \vec{e}_y - 2,2958 \vec{e}_z) \text{ kN}$$

4.12 - STATIKAILAG HATÁROZATLAN SÍKGÖRBE RÚD



Adott:

$$R = 60 \text{ mm}$$

$$d = 24 \text{ mm}$$

$$F_0 = 7 \text{ kN}$$

$$E = 2 \cdot 10^5 \text{ MPa}$$

Feladat:

támazóerők

ismeretlen támasztervek:

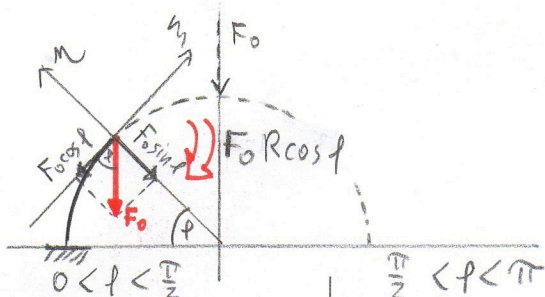
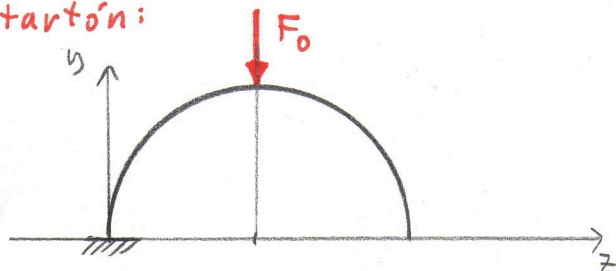
$$F_{Ay}, F_{Az}, M_{Ax}, F_{Cy} \Rightarrow 4 \text{ db}$$

statikai egyenletek:

$$F_y = 0, F_z = 0, M_x = 0 \Rightarrow 3 \text{ db}$$

A tartó statikailag egyserűen határozatlan

eredeti terhelés a határozható tett tartón:



$$N_0 = -F_0 \cos \varphi$$

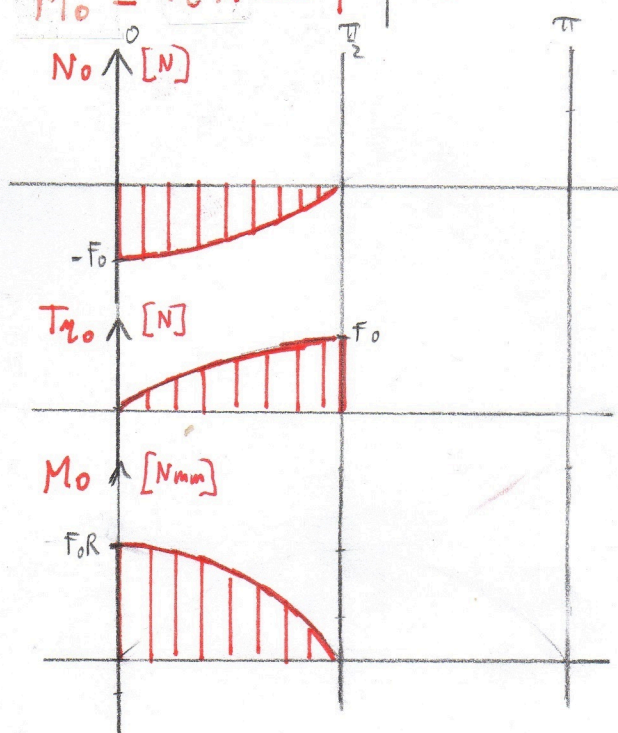
$$N_0 = 0$$

$$T_{\varphi 0} = F_0 \sin \varphi$$

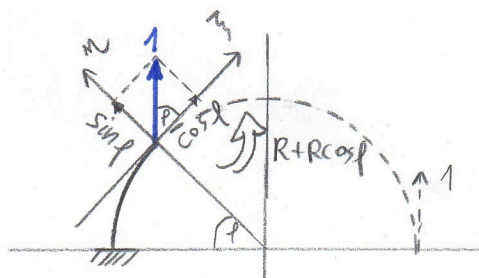
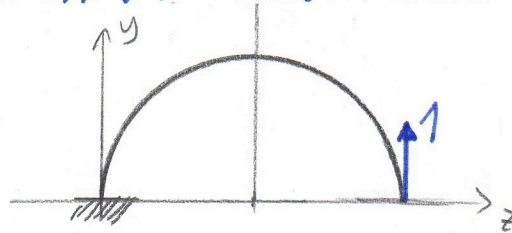
$$T_{\varphi 0} = 0$$

$$M_0 = F_0 R \cos \varphi$$

$$M_0 = 0$$



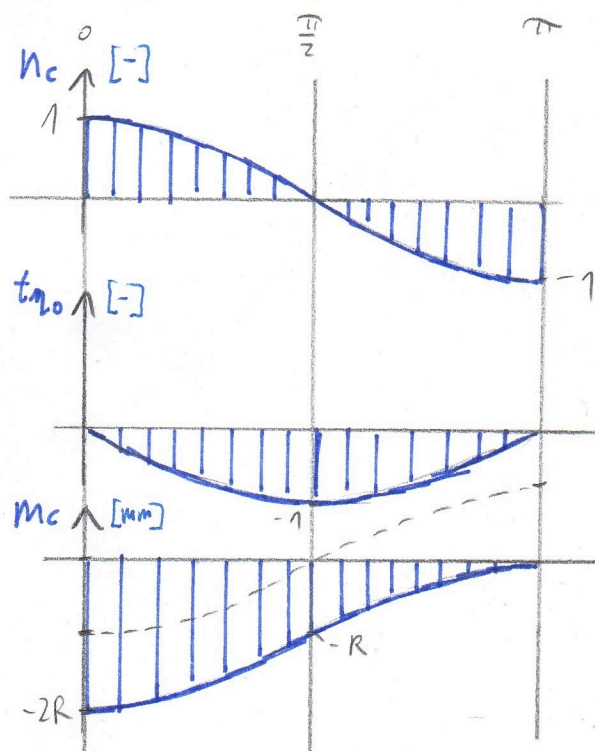
az elhagyott támasztervűnek megfelelő egysegnyi nagyságú terhelés:



$$n_c = \cos \varphi$$

$$t_{\varphi c} = -\sin \varphi$$

$$m_c = -(R + R \cos \varphi) = -R \cos \varphi - R$$



• az alakváltozási energia:

$$U \approx U_N + U_H = \frac{1}{2} \int_{(l)} \frac{N^2}{AE} ds + \frac{1}{2} \int_{(l)} \frac{M_{hx}^2}{I_x E} ds = \frac{1}{2} \int_{\varphi=0}^{\pi} \frac{(N_0 + F_{cy} n_c)^2}{AE} R d\varphi + \frac{1}{2} \int_{\varphi=0}^{\pi} \frac{(M_0 + F_{cy} m_c)^2}{I_x E} R d\varphi$$

• Castigliano - tétel:

$$v_c = 0 = \frac{\partial U}{\partial F_{cy}} = \int_{\varphi=0}^{\pi} \frac{(N_0 + F_{cy} n_c) n_c}{AE} R d\varphi + \int_{\varphi=0}^{\pi} \frac{(M_0 + F_{cy} m_c) m_c}{I_x E} R d\varphi \quad / \cdot E / : R$$

$$0 = \frac{1}{A} \int_{\varphi=0}^{\pi} N_0 n_c d\varphi + F_{cy} \frac{1}{A} \int_{\varphi=0}^{\pi} n_c^2 d\varphi + \frac{1}{I_x} \int_{\varphi=0}^{\pi} M_0 m_c d\varphi + F_{cy} \frac{1}{I_x} \int_{\varphi=0}^{\pi} m_c^2 d\varphi$$

$$F_{cy} = \frac{-\frac{1}{A} \int_{\varphi=0}^{\pi} N_0 n_c d\varphi - \frac{1}{I_x} \int_{\varphi=0}^{\pi} M_0 m_c d\varphi}{\frac{1}{A} \int_{\varphi=0}^{\pi} n_c^2 d\varphi + \frac{1}{I_x} \int_{\varphi=0}^{\pi} m_c^2 d\varphi}$$

• Integrálok kiszámítása:

$$\begin{aligned} \int_{\varphi=0}^{\pi} N_0 n_c d\varphi &= \int_{\varphi=0}^{\frac{\pi}{2}} -F_0 \cos \varphi \cos \varphi d\varphi = -F_0 \int_{\varphi=0}^{\frac{\pi}{2}} \cos^2 \varphi d\varphi = -\frac{F_0}{2} \int_{\varphi=0}^{\frac{\pi}{2}} 1 + \cos 2\varphi d\varphi = \\ &= -\frac{F_0}{2} \left[\varphi + \frac{\sin 2\varphi}{2} \right]_{\varphi=0}^{\frac{\pi}{2}} = -\frac{F_0}{2} \frac{\pi}{2} = -\frac{F_0 \pi}{4} \end{aligned}$$

$$\begin{aligned} \int_{\varphi=0}^{\pi} M_0 m_c d\varphi &= \int_{\varphi=0}^{\frac{\pi}{2}} (F_0 R \cos \varphi) (-R \cos \varphi - R) d\varphi = \int_{\varphi=0}^{\frac{\pi}{2}} -F_0 R^2 \cos^2 \varphi - F_0 R^2 \cos \varphi d\varphi = \\ &= -\frac{F_0 R^2}{2} \int_{\varphi=0}^{\frac{\pi}{2}} 1 + \cos 2\varphi d\varphi - F_0 R^2 \int_{\varphi=0}^{\frac{\pi}{2}} \cos \varphi d\varphi = -\frac{F_0 R^2}{2} \left[\varphi + \frac{\sin 2\varphi}{2} \right]_{\varphi=0}^{\frac{\pi}{2}} - F_0 R^2 [\sin \varphi]_{\varphi=0}^{\frac{\pi}{2}} = \\ &= -\frac{F_0 R^2}{2} \frac{\pi}{2} - F_0 R^2 = -\frac{F_0 R^2 \pi}{4} - F_0 R^2 \end{aligned}$$

$$\int_{\varphi=0}^{\pi} n_c^2 d\varphi = \int_{\varphi=0}^{\pi} \cos^2 \varphi d\varphi = \frac{1}{2} \int_{\varphi=0}^{\pi} 1 + \cos 2\varphi d\varphi = \frac{1}{2} \left[\varphi + \frac{\sin 2\varphi}{2} \right]_{\varphi=0}^{\pi} = \frac{\pi}{2}$$

$$\begin{aligned}
 \int_{\varphi=0}^{\pi} m_c^2 d\varphi &= \int_{\varphi=0}^{\pi} (-R \cos \varphi - R)^2 d\varphi = \int_{\varphi=0}^{\pi} R^2 \cos^2 \varphi + 2R^2 \cos \varphi + R^2 d\varphi = \\
 &= \frac{R^2}{2} \int_{\varphi=0}^{\pi} 1 + \cos 2\varphi d\varphi + 2R^2 \int_{\varphi=0}^{\pi} \cos \varphi d\varphi + \int_{\varphi=0}^{\pi} R^2 d\varphi = \\
 &= \frac{R^2}{2} \left[\varphi + \frac{\sin 2\varphi}{2} \right]_{\varphi=0}^{\varphi=\pi} + 2R^2 [\sin \varphi]_{\varphi=0}^{\varphi=\pi} + [R^2 \varphi]_{\varphi=0}^{\varphi=\pi} = \\
 &= \frac{R^2 \pi}{2} + 0 + R^2 \pi = \frac{3R^2 \pi}{2}
 \end{aligned}$$

• keresztmetszeti jellemzők

$$A = \frac{d^2 \pi}{4} = \frac{24^2 \pi}{4} = 452,389 \text{ mm}^2$$

$$\frac{R}{d/2} = \frac{60}{24/2} = \frac{60}{12} = 5 \quad 3 < 5 \Rightarrow |r \approx |x$$

$$I_x = \frac{d^4 \pi}{64} = \frac{24^4 \pi}{64} = 16286,016 \text{ mm}^4$$

• behelyettesítve F_{cy} képletébe:

$$\begin{aligned}
 F_{cy} &= \frac{-\frac{1}{A} \left(-\frac{F_0 \pi}{4} \right) - \frac{1}{I_x} \left(-\frac{F_0 R^2 \pi}{4} - F_0 R^2 \right)}{\frac{1}{A} \left(\frac{\pi}{2} \right) + \frac{1}{I_x} \left(\frac{3R^2 \pi}{2} \right)} = \\
 &= \frac{\frac{F_0 \pi}{4A} + \frac{F_0 R^2 (\pi + 4)}{4I_x}}{\frac{\pi}{2A} + \frac{3R^2 \pi}{2I_x}} = \\
 &= \frac{\frac{7000 \pi}{4 \cdot 452,389} + \frac{7000 \cdot 60^2 (\pi + 4)}{4 \cdot 16286,016}}{\frac{\pi}{2 \cdot 452,389} + \frac{3 \cdot 60^2 \pi}{2 \cdot 16286,016}} = 2654,929 \text{ N} (\uparrow)
 \end{aligned}$$

• többi + átmetszési koordináta statikai egyenletekből:

$$F_y = 0 = F_{Ay} - F_0 + F_{cy} \Rightarrow F_{Ay} = F_0 - F_{cy} = 7000 - 2654,929 = 4345,071 \text{ N} (\uparrow)$$

$$F_z = 0 = F_{Az}$$

$$M_a = 0 = M_{Ax} + R F_0 - 2R F_{cy} \Rightarrow M_{Ax} = 2R F_{cy} - R F_0 =$$

$$= 2 \cdot 60 \cdot 2654,929 - 60 \cdot 7000 = -101408,32 \text{ Nmm} (5)$$

$$\vec{F}_A = (4345,071 \vec{e}_y) \text{ N} \quad \vec{F}_c = (2654,929 \vec{e}_y) \text{ N} \quad \vec{M}_A = (-101408,32 \vec{e}_x) \text{ Nmm} \quad 3$$