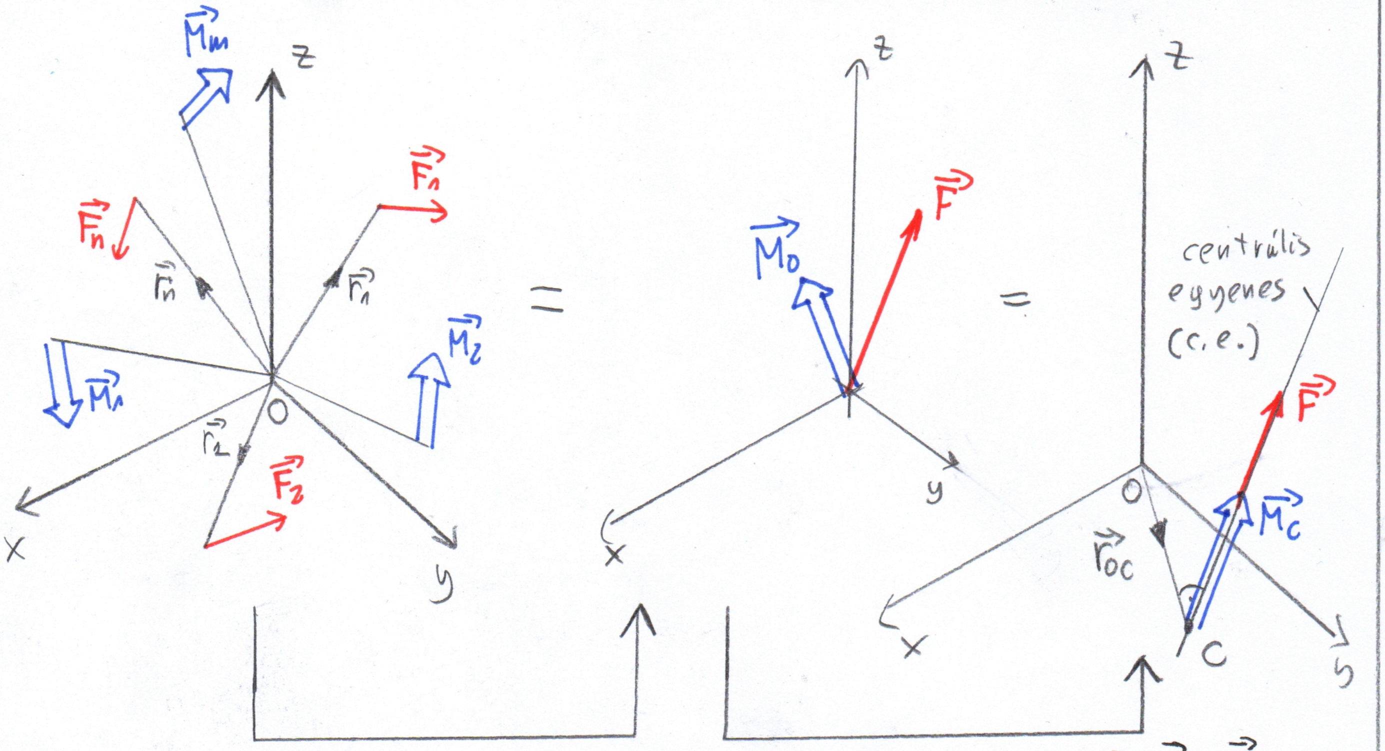
# CENTRÁLIS EGYENES




$$\vec{F} = \sum_{i=1}^{n} \vec{F_i}$$

$$\vec{M_0} = \sum_{i=1}^{n} \vec{r_i} \times \vec{F_i} + \sum_{j=1}^{m} \vec{M_j}$$

c.e. egyenlete: $\vec{a} \times \vec{r} + \vec{b} = \vec{0}$

$$\vec{a} = \vec{F}$$

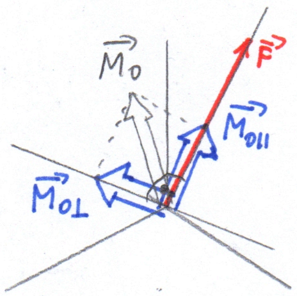
$$\vec{b} = \vec{r_{OC}} \times \vec{F} \quad (= \vec{M_{0\perp}})$$

$$\vec{r_{OC}} = \frac{\vec{F} \times \vec{M_0}}{F^2}$$

$$\vec{M_C} = \vec{M_0} - \vec{M_{0\perp}} = \vec{M_0} - \vec{b}$$

vagy

$$\vec{M_C} = \vec{M_{0\parallel}} = \frac{\vec{F} \cdot \vec{M_0}}{F^2} \vec{F}$$



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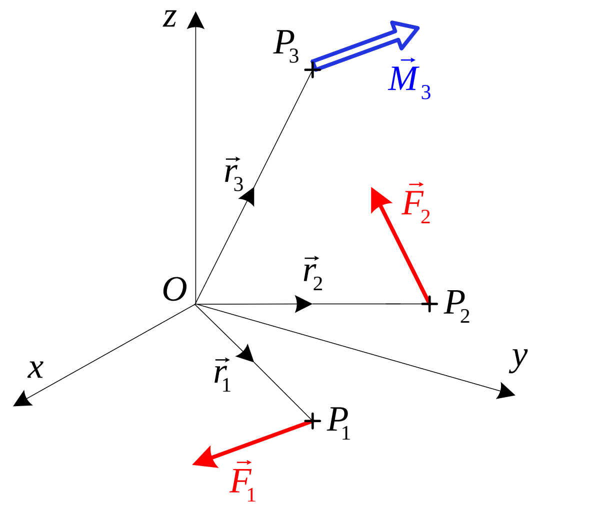
2D esetén: $\vec{M_0} \perp \vec{F} \Rightarrow \vec{M_0} = \vec{M_{0\perp}} = \vec{b} \Rightarrow \vec{a} \times \vec{r} + \vec{b} = \vec{0}$

$$\vec{M_{0\parallel}} = \vec{0}$$

$$\Downarrow$$

$$\vec{F} \times \vec{r} + \vec{M_0} = \vec{0}$$

# 1.1 - CENTRÁLIS EGYENES




Adott:

$\vec{r}_1 = \left(\vec{e}_x + 2\vec{e}_y - \vec{e}_z\right)\mathrm{m}$

$\vec{r}_2 = \left(-2\vec{e}_x + \vec{e}_y\right)\mathrm{m}$

$\vec{r}_3 = \left(-5\vec{e}_x - 3\vec{e}_y - \vec{e}_z\right)\mathrm{m}$

$\vec{F}_1 = \left(2\vec{e}_x - 2\vec{e}_y + \vec{e}_z\right)\mathrm{N}$

$\vec{F}_2 = \left(\vec{e}_x + 3\vec{e}_z\right)\mathrm{N}$

$\vec{M}_3 = \left(-2\vec{e}_x + \vec{e}_y + 4\vec{e}_z\right)\mathrm{Nm}$

Feladat:

a) Centrális egyenes Plücker vektorai ( $\vec{a}, \vec{b}$ )

b) Centrális egyenes tetszőleges pontjának nyomatéka ( $\vec{M}_C$ )

**1. Origóba redukált vektorkettős**

- Eredő erő:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = \left(2\vec{e}_x - 2\vec{e}_y + \vec{e}_z\right) + \left(\vec{e}_x + 3\vec{e}_z\right) = \left(3\vec{e}_x - 2\vec{e}_y + 4\vec{e}_z\right)\mathrm{N}$$

- Origóba redukált nyomaték:

$$\vec{M}_O = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{M}_3$$

$$\vec{r}_1 \times \vec{F}_1 = \left(\vec{e}_x + 2\vec{e}_y - \vec{e}_z\right) \times \left(2\vec{e}_x - 2\vec{e}_y + \vec{e}_z\right) = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 1 & 2 & -1 \\ 2 & -2 & 1 \end{vmatrix} =$$

$$= \vec{e}_x \underbrace{\left(2\cdot 1 - (-2)\cdot(-1)\right)}_{0} - \vec{e}_y \underbrace{\left(1\cdot 1 - 2\cdot(-1)\right)}_{3} + \vec{e}_z \underbrace{\left(1\cdot(-2) - 2\cdot 2\right)}_{-6} = \left(-3\vec{e}_y - 6\vec{e}_z\right)\mathrm{Nm}$$

$$\vec{r}_2 \times \vec{F}_2 = \left(-2\vec{e}_x + \vec{e}_y\right) \times \left(\vec{e}_x + 3\vec{e}_z\right) = -2\cdot 1 \underbrace{\vec{e}_x \times \vec{e}_x}_{\vec{0}} + (-2)\cdot 3 \underbrace{\vec{e}_x \times \vec{e}_z}_{-\vec{e}_y} + 1\cdot 1 \underbrace{\vec{e}_y \times \vec{e}_x}_{\vec{e}_z} + 1\cdot 3 \underbrace{\vec{e}_y \times \vec{e}_z}_{\vec{e}_x} =$$

$$= \left(3\vec{e}_x + 6\vec{e}_y - \vec{e}_z\right)\mathrm{Nm}$$

$$\vec{M}_O = \left(-3\vec{e}_y - 6\vec{e}_z\right) + \left(3\vec{e}_x + 6\vec{e}_y - \vec{e}_z\right) + \left(-2\vec{e}_x + \vec{e}_y + 4\vec{e}_z\right) = \left(\vec{e}_x + 4\vec{e}_y - 3\vec{e}_z\right)\mathrm{Nm}$$

**2. Centrális egyenes Plücker vektorai**

A centrális egyenes Plücker vektoros egyenlete: $\vec{a} \times \vec{r} + \vec{b} = \vec{0}$

2.1 A centrális egyenes irányvektora:

$$\vec{a} = \vec{F} = \underline{\underline{\left(3\vec{e}_x - 2\vec{e}_y + 4\vec{e}_z\right)\mathrm{N}}}$$

2.2 A centrális egyenes origóhoz legközelebbi C pontjának meghatározása:

$$\vec{r}_{OC} = \frac{\vec{F} \times \vec{M}_O}{F^2}$$

$$\vec{F} \times \vec{M}_O = \left(3\vec{e}_x - 2\vec{e}_y + 4\vec{e}_z\right) \times \left(\vec{e}_x + 4\vec{e}_y - 3\vec{e}_z\right) = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 3 & -2 & 4 \\ 1 & 4 & -3 \end{vmatrix} =$$

$$= \vec{e}_x \underbrace{\left((-2)\cdot(-3) - 4\cdot 4\right)}_{-10} - \vec{e}_y \underbrace{\left(3\cdot(-3) - 1\cdot 4\right)}_{-13} + \vec{e}_z \underbrace{\left(3\cdot 4 - 1\cdot(-2)\right)}_{14} = \left(-10\vec{e}_x + 13\vec{e}_y + 14\vec{e}_z\right)\mathrm{N^2 m}$$

$$F^2 = \left(3\vec{e}_x - 2\vec{e}_y + 4\vec{e}_z\right)\cdot\left(3\vec{e}_x - 2\vec{e}_y + 4\vec{e}_z\right) = 3\cdot 3 + (-2)\cdot(-2) + 4\cdot 4 = 29\mathrm{N}^2$$

$$\vec{r}_{OC} = \frac{\left(-10\vec{e}_x + 13\vec{e}_y + 14\vec{e}_z\right)}{29} = \left(-0,345\vec{e}_x + 0,448\vec{e}_y + 0,483\vec{e}_z\right)m$$

2.3 A centrális egyenes irányvektorának nyomatéka az origóra:

$$\vec{b} = \vec{r}_{OC} \times \vec{F} = \frac{1}{29}\left(-10\vec{e}_x + 13\vec{e}_y + 14\vec{e}_z\right) \times \left(3\vec{e}_x - 2\vec{e}_y + 4\vec{e}_z\right) = \frac{1}{29}\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ -10 & 13 & 14 \\ 3 & -2 & 4 \end{vmatrix} =$$

$$= \frac{1}{29}\left[\vec{e}_x \underbrace{\left(13\cdot 4 - (-2)\cdot 14\right)}_{80} - \vec{e}_y \underbrace{\left((-10)\cdot 4 - 3\cdot 14\right)}_{-82} + \vec{e}_z \underbrace{\left((-10)\cdot(-2) - 3\cdot 13\right)}_{-19}\right] =$$

$$= \frac{1}{29}\left(80\vec{e}_x + 82\vec{e}_y - 19\vec{e}_z\right) = \underline{\underline{\left(2,759\vec{e}_x + 2,828\vec{e}_y - 0,655\vec{e}_z\right)\text{Nm}}}$$

**3. A centrális egyenes tetszőleges pontjának nyomatéka**

egyik módszerrel:

$$\vec{M}_C = \vec{M}_O - \vec{M}_{O\perp} = \vec{M}_O - \vec{b} = \left(\vec{e}_x + 4\vec{e}_y - 3\vec{e}_z\right) - \left(2,759\vec{e}_x + 2,828\vec{e}_y - 0,655\vec{e}_z\right) =$$

$$= \left(-1,759\vec{e}_x + 1,172\vec{e}_y - 2,345\vec{e}_z\right)\text{Nm}$$

másik módszerrel:

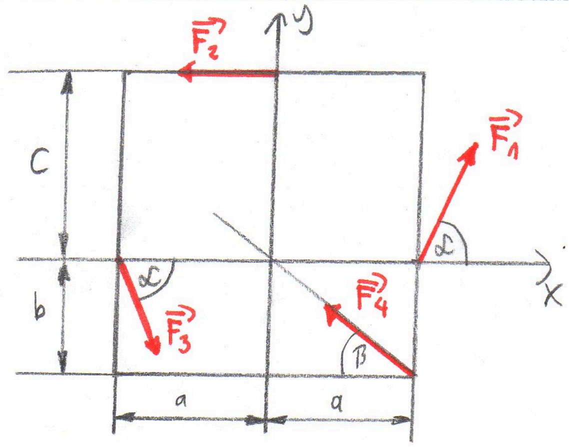
$$\vec{M}_C = \vec{M}_{O\|} = \frac{\vec{F}\cdot\vec{M}_O}{F^2}\vec{F}$$

$$\vec{F}\cdot\vec{M}_O = \left(3\vec{e}_x - 2\vec{e}_y + 4\vec{e}_z\right)\cdot\left(\vec{e}_x + 4\vec{e}_y - 3\vec{e}_z\right) = 3\cdot 1 - 2\cdot 4 + 4\cdot(-3) = -17\text{N}^2\text{m}$$

$$F^2 = \vec{F}\cdot\vec{F} = 29\text{N}^2$$

$$\vec{M}_C = -\frac{17}{29}\left(3\vec{e}_x - 2\vec{e}_y + 4\vec{e}_z\right) = \left(-1,759\vec{e}_x + 1,172\vec{e}_y - 2,345\vec{e}_z\right)\text{Nm}$$

# CENTRÁLIS EGYENES (2D)




**Adott:**

$\vec{F_1} = \vec{F_2} = \vec{F_3} = 10\,kN \quad \vec{F_4} = 50\,kN$

$\alpha = 60^\circ \quad a = 0{,}4\,m \quad b = 0{,}3\,m \quad c = 0{,}5\,m$

**Feladat:**

a) O pontra redukált vektorkettős

b) c.e. egyenesének Plücker vektorai

→ irányvektor meghatározása ($\vec{a}$)

→ C pont meghatározása ($\vec{r}_{OC}$)

→ irányvektor nyomatékának meghatározása ($\vec{b}$)

c) c.e. tetszőleges pontjának nyomatéka

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a) O pontra redukált vektorkettős

$$\vec{F} = \sum_{i=1}^{4} \vec{F_i} = F_x \vec{e}_x + F_y \vec{e}_y \qquad \cos\beta = \frac{0{,}4}{0{,}5} = 0{,}8 \quad \sin\beta = \frac{0{,}3}{0{,}5} = 0{,}6$$

2, $\sqrt{3}$, 1 $\qquad \cos 60^\circ = 0{,}5 \quad \sin 60^\circ = \frac{\sqrt{3}}{2}$

$$F_x = \sum_{i=1}^{4} F_{ix} = F_1\cos\alpha - F_2 + F_3\cos\alpha - F_4\cos\beta = 10\cdot 0{,}5 - 10 + 10\cdot 0{,}5 - 50\cdot 0{,}8 = -40\,kN$$

$$F_y = \sum_{i=1}^{4} F_{iy} = F_1\sin\alpha - F_3\sin\alpha + F_4\sin\beta = 10\frac{\sqrt{3}}{2} - 10\frac{\sqrt{3}}{2} + 50\cdot 0{,}6 = 30\,kN$$

$$\vec{F} = (-40\vec{e}_x + 30\vec{e}_y)\,kN$$

$$M_{Oz} = a F_1 \sin\alpha + c F_2 + a F_3 \sin\alpha = 0{,}4\cdot 10\cdot\frac{\sqrt{3}}{2} + 0{,}5\cdot 10 + 0{,}4\cdot 10\cdot \frac{\sqrt{3}}{2} = 11{,}93\,kNm$$

$$\vec{M}_O = M_z \vec{e}_z = (11{,}93\,\vec{e}_z)\,kNm$$

(síkbeli E.R. esetén $\vec{F} \perp \vec{M}_O \Rightarrow \vec{F}\cdot\vec{M}_O = 0$)

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b) • c.e. irányvektora: $\qquad \vec{a}\times\vec{v} + \vec{b} = \vec{0}$

$$\vec{a} = \vec{F} = (-40\vec{e}_x + 30\vec{e}_y)\,N$$

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• c.e. C pontjának meghatározása

$$\vec{r}_{OC} = \frac{\vec{F}\times\vec{M}_O}{F^2}$$

$$\vec{F}\times\vec{M}_O = (-40\vec{e}_x + 30\vec{e}_y)\times 11{,}93\vec{e}_z = -40\cdot 11{,}93\,\overbrace{\vec{e}_x\times\vec{e}_z}^{-\vec{e}_y} + 30\cdot 11{,}93\,\overbrace{\vec{e}_y\times\vec{e}_z}^{\vec{e}_x} = (357{,}9\,\vec{e}_x + 477{,}2\,\vec{e}_y)\,N^2m$$

$$F^2 = \vec{F}\cdot\vec{F} = (-40\vec{e}_x + 30\vec{e}_y)\cdot(-40\vec{e}_x + 30\vec{e}_y) = -40\cdot(-40) + 30\cdot 30 = 2500\,N^2$$

$$\vec{r}_{OC} = \frac{\vec{F}\times\vec{M}_O}{F^2} = \frac{357{,}9\,\vec{e}_x + 477{,}2\,\vec{e}_y}{2500} = (0{,}143\,\vec{e}_x + 0{,}191\,\vec{e}_y)\,m$$

• irányvektor nyomatékának meghatározása

$e_z \; e_x \; e_y$ +

$$\vec b = \vec r_{0C} \times \vec F = (0{,}143\vec e_x + 0{,}191\vec e_y)\times(-40\vec e_x + 30\vec e_y) =$$

$$= \underbrace{0{,}143\cdot 30}\,\underbrace{\vec e_x\times\vec e_y}_{\vec e_z} - \underbrace{0{,}191\cdot 40}\,\underbrace{\vec e_y\times\vec e_x}_{-\vec e_z} = (11{,}93\,\vec e_z)\ \mathrm{Nm}$$

a plücker vektoros egyenlet:

$$\vec a \times \vec r + \vec b = \vec 0$$

$$(-40\vec e_x + 30\vec e_y)\times(x\vec e_x + y\vec e_y) + 11{,}93\vec e_z = \vec 0$$

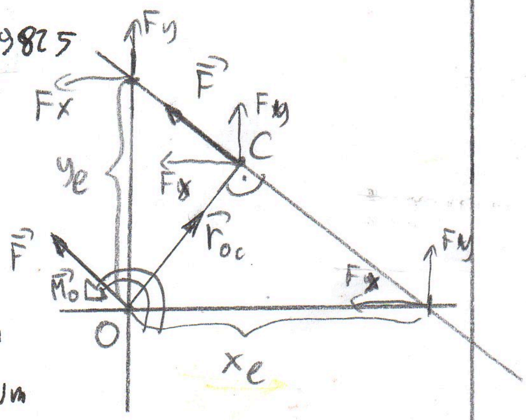
$$-40x\underbrace{\vec e_x\times\vec e_x}_{\vec 0} - 40y\underbrace{\vec e_x\times\vec e_y}_{\vec e_z} + 30x\underbrace{\vec e_y\times\vec e_x}_{-\vec e_z} + 30y\underbrace{\vec e_y\times\vec e_y}_{\vec 0} + 11{,}93\vec e_z = \vec 0$$

$$-40y\,\vec e_z - 30x\,\vec e_z + 11{,}93\,\vec e_z = \vec 0 \quad /\cdot\vec e_z$$

$$-40y - 30x + 11{,}93 = 0$$

$$-40y = 30x - 11{,}93$$

$$y = -0{,}75x + 0{,}29825$$




metszéspontok:

$x = 0 \Rightarrow y_e = 0{,}29825$ m

$y = 0 \Rightarrow x_e = \frac{-0{,}29825}{-0{,}75} = 0{,}397\dot{6}$ m

ellenőrzés:

$M_{0z} = x_e F_y = 0{,}397\dot{6}\cdot 30 = 11{,}93$ Nm

$M_{0z} = -y_e F_x = -0{,}29825\cdot(-40) = 11{,}93$ Nm

c) c.e. tetszőleges pontjára számított nyomaték:

$$\vec M_C = \vec M_0 + \vec F\times\vec r_{0C} = \vec M_0 - \underbrace{\vec r_{0C}\times\vec F}_{\vec b = \vec M_\perp} = 11{,}93\vec e_z - 11{,}93\vec e_z = \vec 0$$

vagy $\vec F \perp \vec M_0$

$$\vec M_C = \vec M_{0\parallel} = \frac{\vec F\cdot\vec M_0}{F^2}\vec F = \frac{0}{F^2}\vec F = \vec 0$$