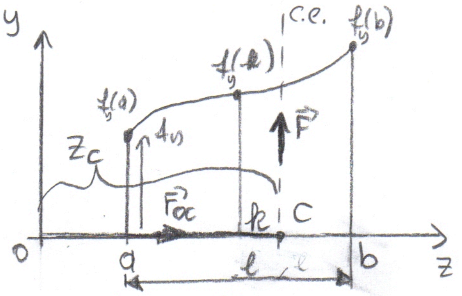
MEGOSZLÓ ERŐRENDSZEREK

① egyenes vonal mentén megoszló párhuzamos E.R.

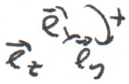


• E.R. sűrűségvektora: $\vec f_y(z) = \dfrac{d\vec F}{dz}$

$$\Downarrow$$

$$d\vec F = \vec f_y(z)\,dz$$

$$\underbrace{\int_{(l)} d\vec F}_{\vec F} = \int_{(l)} \vec f_y(z)\,dz$$



• eredő erő: $\vec F = \int_{(l)} \vec f_y(z)\,dz = \int_{(l)} f_y(z)\,dz\,\vec e_y = \overbrace{\frac{l}{6}\left(f_y(a)+4f_y(k)+f_y(b)\right)}^{F}\vec e_y$

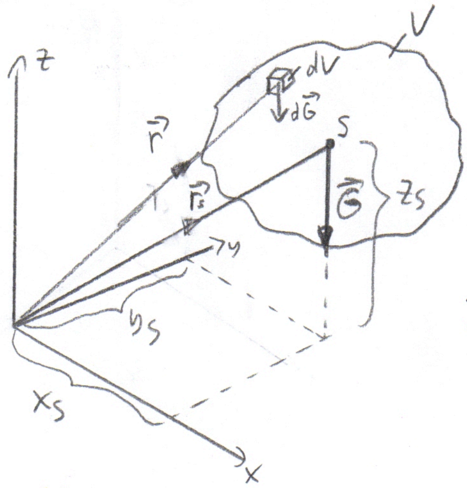
• nyomaték origóra: $\vec M_O = \int_{(l)} \vec r\times d\vec F = \int_{(l)} \vec r\times \vec f_y(z)\,dz = \int_{(l)} z\vec e_z\times f_y(z)\vec e_y\,dz =$

$$= -\int_{(l)} z\,f_y(z)\,dz\,\vec e_x = \underbrace{-\frac{l}{6}\left(a\,f_y(a)+4k\,f_y(k)+b\,f_y(b)\right)}_{M_{Ox}}\vec e_x$$

• c.e. C pontja: $\vec r_{OC} = \dfrac{\vec F\times\vec M_O}{F^2} = \dfrac{F\vec e_y\times M_{Ox}\vec e_x}{F^2} = \dfrac{-\cancel{F}M_{Ox}\vec e_z}{F^{\cancel{2}}} = -\underbrace{\dfrac{M_{Ox}}{F}}_{z_C}\vec e_z$

② súlypontszámítás

Súlypont: test térfogatán megoszló súlyerőrendszer S középpontja



$$\vec r_S = \frac{\int_{(V)} \vec r\,dG}{G} = \frac{\int_{(V)} \vec r\,\overbrace{\cancel{\rho g}\,dV}^{dG}}{\underbrace{\cancel{\rho g}V}_{G}} = \frac{\int_{(V)} \vec r\,dV}{V}$$

A súlypont helyvektorának koordinátái:

$$x_S = \frac{\int_{(V)} x\,dV}{V} \qquad y_S = \frac{\int_{(V)} y\,dV}{V} \qquad z_S = \frac{\int_{(V)} z\,dV}{V}$$

$$\vec r_S = x_S\vec e_x + y_S\vec e_y + z_S\vec e_z$$

hasonlóan:

- síkidomra xy síkban

$$x_S = \frac{\int_{(A)} x\,dA}{A} \qquad y_S = \frac{\int_{(A)} y\,dA}{A}$$

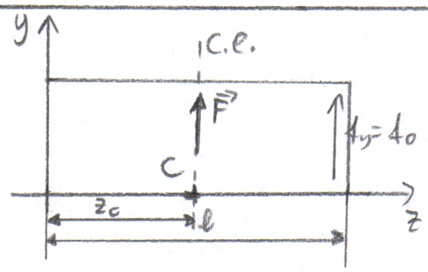
$$\vec r_S = x_S\vec e_x + y_S\vec e_y$$

- vonalra

ívkoordináta

$$\vec r_S = \frac{\int_{(l)} \vec r\,ds}{l}$$

VONAL MENTÉN MEGOSZLÓ ERŐRENDSZER



• eredő erő: A: $F=\int_{z=0}^{\ell} f_y(z)dz=\int_{z=0}^{\ell} f_0\,dz=f_0\int_{z=0}^{\ell}dz=f_0\ell$

S: $F=\int_{z=0}^{\ell} f_y(z)dz=\frac{\ell}{6}\left(f_0+4f_0+f_0\right)=\ell f_0$

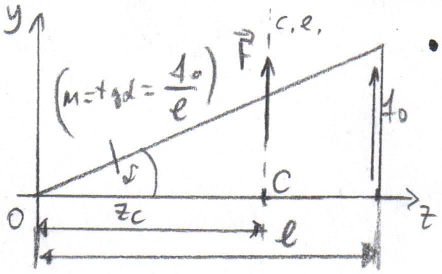
• nyomaték O pontra: A: $M_{Ox}=-\int_{z=0}^{\ell} z f_y(z)dz=-\int_{z=0}^{\ell} z f_0\,dz=-f_0\int_{z=0}^{\ell} z\,dz=$

$=-f_0\left[\frac{z^2}{2}\right]_0^{\ell}=-f_0\frac{\ell^2}{2}$

S: $M_{Ox}=-\int_{z=0}^{\ell} z f_y(z)dz=-\frac{\ell}{6}\left(0\cdot f_0+\underbrace{4\cdot\frac{\ell}{2}f_0+\ell f_0}_{3\ell f_0}\right)=-f_0\frac{\ell^2}{2}$

• c.e. C pontja:

$z_C=-\frac{M_{Ox}}{F}=-\frac{-\cancel{f_0}\frac{\ell^{\cancel{2}}}{2}}{\cancel{\ell}\cancel{f_0}}=\frac{\ell}{2}$



• eredő erő: A: $F=\int_{z=0}^{\ell} f_y(z)dz=\int_{z=0}^{\ell}\frac{f_0}{\ell}z\,dz=\frac{f_0}{\ell}\int_{z=0}^{\ell} z\,dz=\frac{f_0}{\ell}\left[\frac{z^2}{2}\right]_0^{\ell}=\frac{f_0\ell}{2}$

S: $F=\int_{z=0}^{\ell} f_y(z)dz=\frac{\ell}{6}\left(0+4\frac{f_0}{2}+f_0\right)=\frac{f_0\ell}{2}$

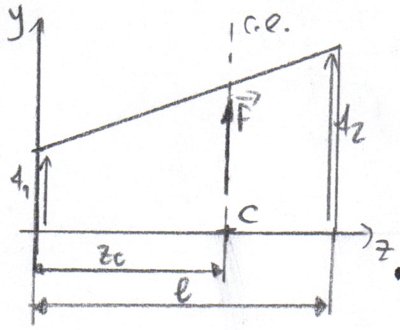
• nyomaték O pontra: A: $M_{Ox}=-\int_{z=0}^{\ell} z f_y(z)dz=-\int_{z=0}^{\ell} z\frac{f_0}{\ell}z\,dz=-\frac{f_0}{\ell}\int_{z=0}^{\ell} z^2dz=$

$=-\frac{f_0}{\ell}\left[\frac{z^3}{3}\right]_0^{\ell}=-\frac{f_0\ell^2}{3}$

S: $M_{Ox}=-\int_{z=0}^{\ell} z f_y(z)dz=-\frac{\ell}{6}\left(0\cdot 0+\underbrace{\cancel{4}\cdot\frac{\ell}{\cancel{2}}\cdot\frac{f_0}{\cancel{2}}+\ell\cdot f_0}_{2\ell f_0}\right)=-\frac{f_0\ell^2}{3}$

• c.e. C pontja:

$z_C=-\frac{M_{Ox}}{F}=-\frac{-\frac{f_0\ell^2}{3}}{\frac{\cancel{f_0}\ell}{2}}=\frac{2}{3}\ell$



• eredő erő: S: $F=\int_{z=0}^{\ell} f_y(z)dz=\frac{\ell}{6}\left(f_1+4\frac{f_1+f_2}{2}+f_2\right)=$

$=\frac{\ell}{6}\left(f_1+2f_1+2f_2+f_2\right)=\frac{\ell}{6}3(f_1+f_2)=\ell\frac{f_1+f_2}{2}$

• nyomaték O pontra: S: $M_{Ox}=-\int_{z=0}^{\ell} z f_y(z)dz=-\frac{\ell}{6}\left(0\cdot f_1+\cancel{4}\cdot\frac{\ell}{\cancel{2}}\cdot\frac{f_1+f_2}{\cancel{2}}+\ell\cdot f_2\right)=$

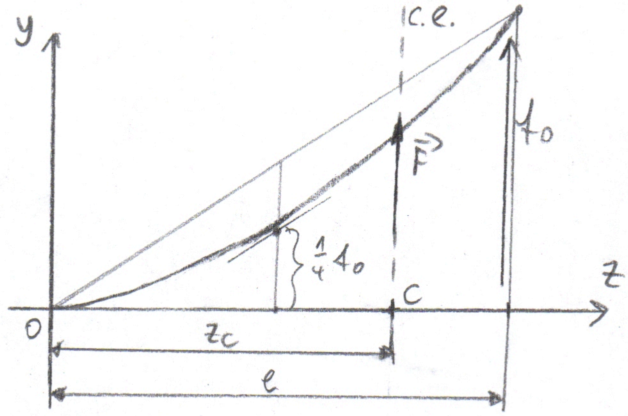
$=-\frac{\ell}{6}\left(\ell f_1+2\ell f_2\right)=-\frac{\ell^2}{6}(f_1+2f_2)$

c.e. C pontja:

$z_C=-\frac{M_{Ox}}{F}=-\frac{-\frac{\ell^2}{6}(f_1+2f_2)}{\ell\frac{f_1+f_2}{2}}=$

$=\frac{\cancel{2}}{\cancel{\ell}(f_1+f_2)}\cdot\frac{\ell^{\cancel{2}}(f_1+2f_2)}{\cancel{6}_{3}}=\frac{\ell}{3}\,\frac{f_1+2f_2}{f_1+f_2}$

VONAL MENTÉN MEGOSZLÓ ERŐRENDSZER



• eredő erő:

$$F=\int_{z=0}^{\ell} f_y(z)\,dz=\frac{\ell}{6}\Big(\underbrace{0+4\tfrac{1}{4}f_0+f_0}_{2f_0}\Big)=$$

$$=\frac{f_0\,\ell}{3}$$

• nyomaték 0-ra

$$M_{0x}=-\int_{z=0}^{\ell} z\,f_y(z)\,dz=$$

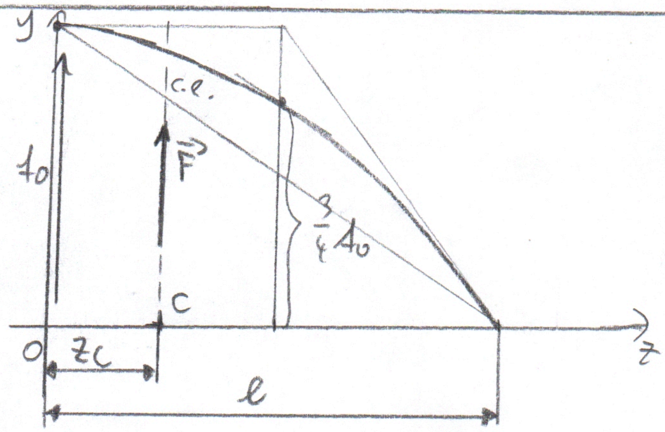
$$=-\frac{\ell}{6}\Big(\underbrace{0\cdot 0+4\cdot\frac{\ell}{2}\,\frac{1}{4}f_0+\ell f_0}_{\frac{3}{2}\ell f_0}\Big)=$$

$$=-\frac{\ell}{\cancel{6}_{2}}\,\frac{\cancel{3}}{2}\,\ell f_0=-\frac{\ell^2 f_0}{4}$$

• c.e. C pontja:

$$z_C=-\frac{M_{0x}}{F}=-\frac{-\frac{\ell^2 f_0}{4}}{\frac{f_0\ell}{3}}=$$

$$=\frac{3}{\cancel{f_0}\cancel{\ell}}\,\frac{\cancel{\ell^2}\cancel{f_0}}{4}=\frac{3}{4}\ell$$



• eredő erő:

$$F=\int_{z=0}^{\ell} f_y(z)\,dz=\frac{\ell}{6}\Big(\underbrace{f_0+4\tfrac{3}{4}f_0+0}_{4f_0}\Big)=$$

$$\frac{2}{3}\ell f_0$$

• nyomaték 0-ra

$$M_{0x}=-\int_{z=0}^{\ell} z\,f_y(z)\,dz=$$

$$=-\frac{\ell}{6}\Big(\underbrace{0f_0+4\frac{\ell}{2}\,\frac{3}{4}f_0+\ell\cdot 0}_{\frac{3}{2}\ell f_0}\Big)=$$

$$=-\frac{\ell}{\cancel{6}_{2}}\,\frac{\cancel{3}}{2}\,\ell f_0=-\frac{\ell^2 f_0}{4}$$

• c.e. C pontja:

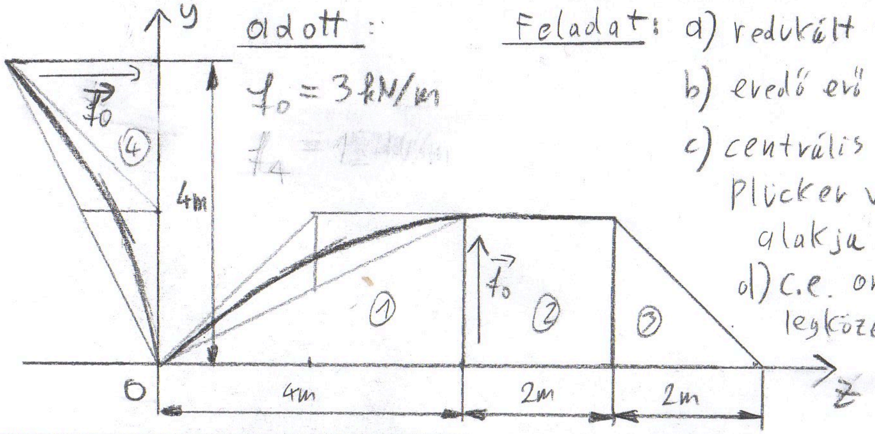
$$z_C=-\frac{M_{0x}}{F}=-\frac{-\frac{\ell^2 f_0}{4}}{\frac{2}{3}\ell f_0}=$$

$$=\frac{3}{2\cancel{\ell}\cancel{f_0}}\cdot\frac{\cancel{\ell^2}\cancel{f_0}}{4}=\frac{3}{8}\ell$$

adott:

$f_0 = 3\,kN/m$

Feladat:
a) redukált vektorkettős
b) eredő erő hatásvonala
c) centrális egyenes Plücker vektoros alakja
d) c.e. origóhoz legközelebbi C pontja



a) redukált vektorkettős

- a vízszintes megoszló erőrendszer eredője:

$$F_{1y} = \int_0^4 f_{1y}(z)\,dz = \frac{4}{6}\left(0 + 4\cdot\frac{3}{4}\cdot 3 + 3\right) = \frac{4}{6}\cdot 12 = 8\,kN$$

$$F_{2y} = \int_4^6 f_{2y}(z)\,dz = \frac{2}{6}\left(3 + 4\cdot 3 + 3\right) = \frac{2}{6}\cdot 18 = 6\,kN$$

$$F_{3y} = \int_6^8 f_{3y}(z)\,dz = \frac{2}{6}\left(3 + 4\cdot\frac{1}{2}\,3 + 0\right) = \frac{2}{6}\cdot 9 = 3\,kN$$

$$\vec{F}' = (F_{1y} + F_{2y} + F_{3y})\,\vec{e}_y = (8 + 6 + 3)\,\vec{e}_y = (17\,\vec{e}_y)\,kN$$

- a Függőleges megoszló erőrendszer eredője

$$F_{4z} = \int_0^4 f_{4z}(y)\,dy = \frac{4}{6}\left(0 + 4\cdot\frac{1}{4}\,3 + 3\right) = \frac{4}{6}\cdot 6 = 4\,kN$$

$$\vec{F}'' = F_{4z}\,\vec{e}_z = (4\,\vec{e}_z)\,kN$$

- az eredő erő:

$$\vec{F} = \vec{F}' + \vec{F}'' = (17\,\vec{e}_y + 4\,\vec{e}_z)\,kN$$

- a vízszintes megoszló erőrendszer nyomatéka az O pontra:

$$M_{ox1} = \int_0^4 z f_{1y}(z)\,dz = -\frac{4}{6}\left(0\cdot 0 + 4\cdot 2\cdot\frac{3}{4}\cdot 3 + 4\cdot 3\right) = -\frac{4}{6}\cdot 30 = -20\,kNm$$

$$M_{ox2} = \int_4^6 z f_{2y}(z)\,dz = -\frac{2}{6}\left(4\cdot 3 + 4\cdot 5\cdot 3 + 6\cdot 3\right) = -\frac{2}{6}\,90 = -30\,kNm$$

$$M_{ox3} = \int_6^8 z f_{3y}(z)\,dz = -\frac{2}{6}\left(6\cdot 3 + 4\cdot 7\cdot\frac{1}{2}\cdot 3 + 8\cdot 0\right) = -\frac{2}{6}\cdot 60 = -20\,kNm$$

$$\vec{M}_O' = (M_{ox1} + M_{ox2} + M_{ox3})\,\vec{e}_x = (-20 - 30 - 20)\,\vec{e}_x = (-70\,\vec{e}_x)\,kNm$$

- a Függőleges megoszló erőrendszer nyomatéka az O pontra:

$$M_{ox4} = \int_0^4 y f_{4z}(y)\,dz = \frac{4}{6}\left(0\cdot 0 + 4\cdot 2\cdot\frac{1}{4}\,3 + 4\cdot 3\right) = \frac{4}{6}\cdot 18 = 12\,kNm$$

$$\vec{M}_O'' = M_{ox4}\,\vec{e}_x = (12\,\vec{e}_x)\,kNm$$

- az origóra számított nyomaték:

$$\vec{M}_O = \vec{M}_O' + \vec{M}_O'' = -70\,\vec{e}_x + 12\,\vec{e}_x = (-58\,\vec{e}_x)\,kNm$$

b) eredő erő hatásvonala:

- a vízszintes megoszló e.r. eredőjének hatásvonala:

$$M_{ox}' = -z_e' F' \Rightarrow z_e' = -\frac{M_{ox}'}{F_y'} = -\frac{-70}{17} = 4{,}12\,m$$

- a Függőleges megoszló e.r. eredőjének hatásvonala:

$$M_{ox}'' = y_e'' F'' \Rightarrow y_e'' = \frac{M_{ox}''}{F_z''} = \frac{12}{4} = 3\,m$$

- az eredő erő hatásvonalának z, y metszéspontjai:

$$M_{ox} = -z_e F_y \Rightarrow z_e = -\frac{M_{ox}}{F_y} = -\frac{-58}{17} = 3{,}41\,m$$

$$M_{ox} = y_e F_z \Rightarrow y_e = \frac{M_{ox}}{F_z} = \frac{-58}{4} = -14{,}5\,m$$

c) centrális egyenes:

$$\vec{a} \times \vec{r} + \vec{b} = \vec{0}$$

$$\vec{a} = \vec{F} = \left(17\vec{e}_y + 4\vec{e}_z\right) kN$$

$$\vec{b} = \vec{M}_0 - \underbrace{\vec{M}_{0II}}_{=\vec{0}} = \vec{M}_0 = \left(-58\vec{e}_x\right) kNm$$

d) C pont

$$\vec{r}_C = \frac{\vec{F} \times \vec{M}_0}{F^2} =$$

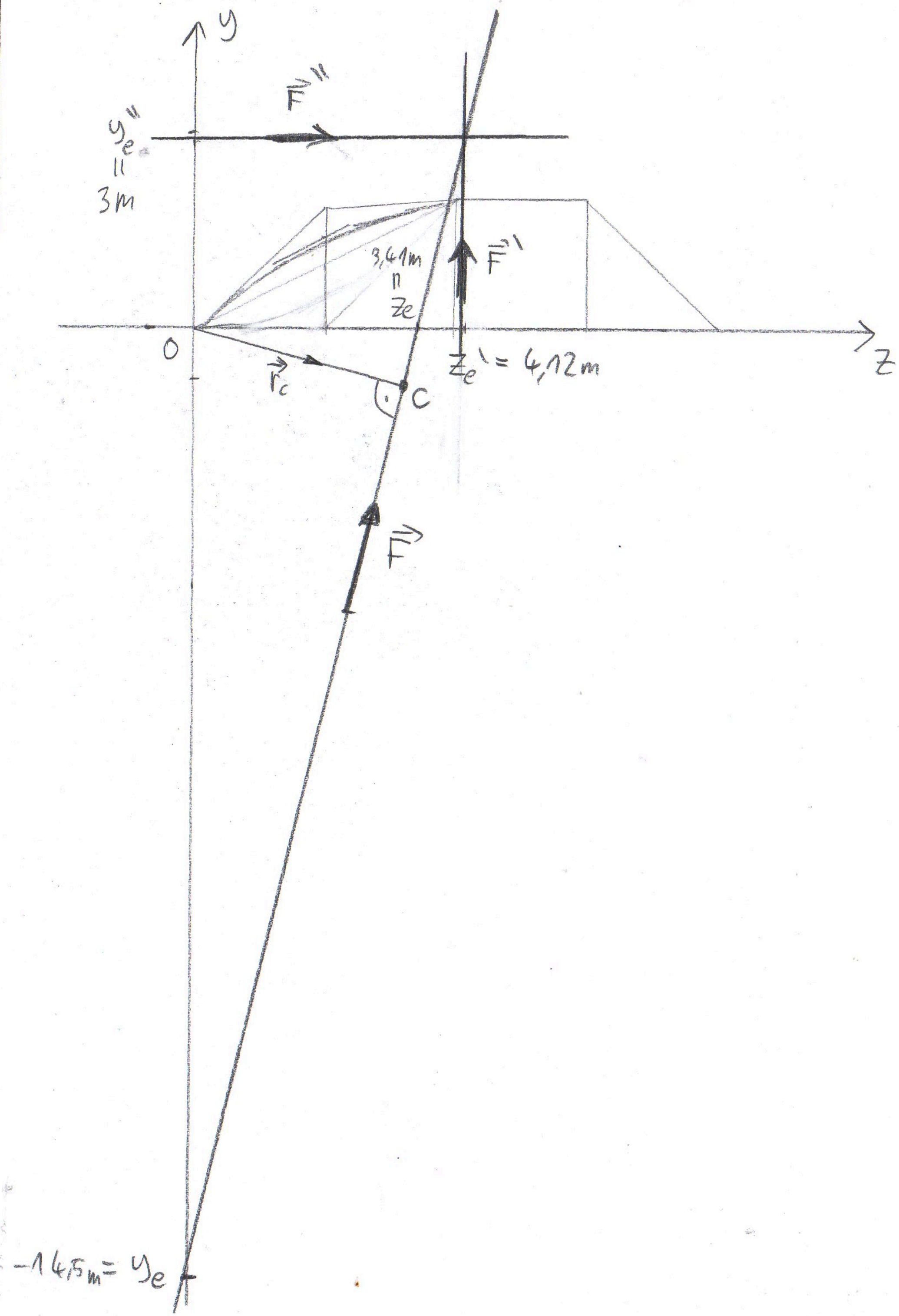
$$\vec{F} \times \vec{M}_0 = \begin{vmatrix} \overset{+}{\vec{e}_x} & \overset{-}{\vec{e}_y} & \overset{+}{\vec{e}_z} \\ 0 & 17 & 4 \\ -58 & 0 & 0 \end{vmatrix} = \vec{e}_x(17 \cdot 0 - 4 \cdot 0) - \vec{e}_y(0 \cdot 0 - (-58) \cdot 4) +$$

$$+ \vec{e}_z(0 \cdot 0 - (-58) \cdot 17) =$$

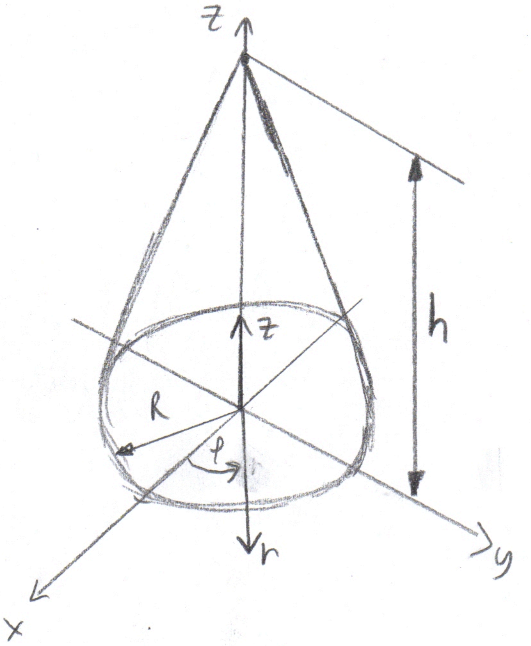
$$= \left(-232\vec{e}_y + 986\vec{e}_z\right) kN^2 m$$

$$F^2 = 17^2 + 4^2 = 305\ kN^2$$

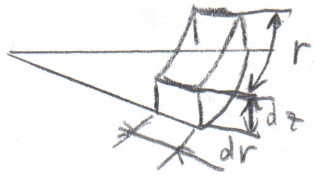
$$= \frac{-232\vec{e}_y + 986\vec{e}_z}{305} = \left(-0{,}76\vec{e}_y + 3{,}23\vec{e}_z\right) m$$



TEST SÚLYPONTJA



• elemi térfogat henger k.r.-ben:



$dV = r\,d\varphi\,dz\,dr$

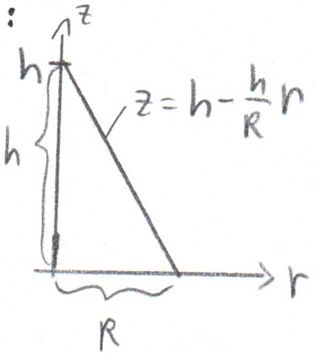
• integrálási határok:

$\varphi \to 0 \to 2\pi$

$r \to 0 \to R$

$z \to$

$z \to 0 \to h - \frac{h}{R} r$



$$V = \int_V dV = \int_{r=0}^{R} \int_{z=0}^{h-\frac{h}{R}r} \int_{\varphi=0}^{2\pi} r\,d\varphi\,dz\,dr = 2\pi \int_{r=0}^{R} \int_{z=0}^{h-\frac{h}{R}r} r\,dz\,dr =$$

$$= 2\pi \int_{r=0}^{R} r \left[z \right]_0^{h-\frac{h}{R}r} dr = 2\pi \int_{r=0}^{R} r\left(h - \frac{h}{R} r\right) dr =$$

$$= 2\pi \int_{r=0}^{R} -\frac{h}{R} r^2 + h r\,dr = 2\pi \left[-\frac{h}{R}\frac{r^3}{3} + h\frac{r^2}{2} \right]_0^R = 2\pi \left(-\frac{h}{R}\frac{R^{\cancel{3}\,2}}{3} + h\frac{R^2}{2} \right) =$$

$$= 2\pi \left(-\frac{hR^2}{3} + \frac{hR^2}{2} \right) = -\frac{2\pi h R^2}{3} + \pi h R^2 = \frac{\pi h R^2}{3}$$

$$\int_{(V)} \vec{r}\,dV = \int_{(V)} r\cos\varphi\,\vec{e}_x + r\sin\varphi\,\vec{e}_y + z\,\vec{e}_z\,dV = \int_{r=0}^{R} \int_{z=0}^{h-\frac{h}{R}r} \int_{\varphi=0}^{2\pi} r\cos\varphi\,\vec{e}_x + r\sin\varphi\,\vec{e}_y + z\,\vec{e}_z \overbrace{r\,d\varphi\,dz\,dr}^{dV} =$$

$$= 2\pi \int_{r=0}^{R} \int_{z=0}^{h-\frac{h}{R}r} z\,r\,dz\,dr\,\vec{e}_z = 2\pi \int_{r=0}^{R} r \left[\frac{z^2}{2} \right]_0^{h-\frac{h}{R}r} dr = 2\pi \int_{r=0}^{R} r \left(\frac{\left(h - \frac{h}{R} r\right)^2}{2} \right) dr\,\vec{e}_z =$$

$$= 2\pi \int_{r=0}^{R} r \left(\frac{h^2}{2} - \frac{h^2}{R} r + \frac{h^2}{2R^2} r^2 \right) dr\,\vec{e}_z = 2\pi \int_{r=0}^{R} \frac{h^2}{2R^2} r^3 - \frac{h^2}{R} r^2 + \frac{h^2}{2} r\,dr\,\vec{e}_z =$$

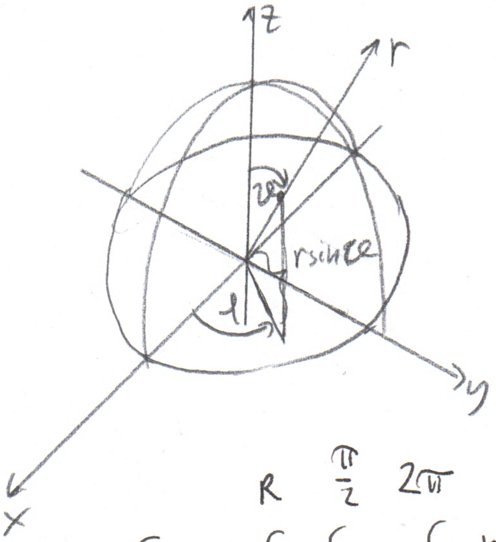
$$= 2\pi \cdot \left[\frac{h^2}{2R^2}\frac{r^4}{4} - \frac{h^2}{R}\frac{r^3}{3} + \frac{h^2}{2}\frac{r^2}{2} \right]_0^R \vec{e}_z = 2\pi \left(\frac{h^2 R^{\cancel{4}\,2}}{8\cancel{R^2}} - \frac{h^2 R^{\cancel{3}\,2}}{3\cancel{R}} + \frac{h^2 R^{\cancel{3}}}{4} \right) \vec{e}_z$$

$$= 2\pi R^2 h^2 \left(\frac{1}{8} - \frac{1}{3} + \frac{1}{4} \right) \vec{e}_z = 2\pi R^2 h^2 \frac{1}{24} \vec{e}_z = \frac{h^2 R^2 \pi}{12} \vec{e}_z$$

$$\vec{r}_s = \frac{\int_{(V)} \vec{r}\,dV}{V} = \frac{h^{\cancel{2}} \cancel{R^2} \cancel{\pi}}{4 \cancel{12}} \cdot \frac{\cancel{3}}{\cancel{\pi h R^2}} \vec{e}_z = \boxed{\frac{h}{4} \vec{e}_z}$$

TEST SÚLYPONTJA

$2\cos x \sin x = \sin 2x$



$$V=\int_{(V)} dV=\int_{r=0}^{R}\int_{\vartheta=0}^{\frac{\pi}{2}}\int_{\varphi=0}^{2\pi} r^2\sin\vartheta \, d\varphi \, d\vartheta \, dr = 2\pi\int_{r=0}^{R}\int_{\vartheta=0}^{\frac{\pi}{2}} r^2 \sin\vartheta \, d\vartheta \, dr =$$

$$=2\pi\int_{r=0}^{R} r^2\left[-\cos\vartheta\right]_0^{\frac{\pi}{2}} dr = 2\pi\int_{r=0}^{R} r^2 \overbrace{\left(0-(-1)\right)}^{1} dr = 2\pi\left[\frac{r^3}{3}\right]_0^{R} = \frac{2\pi R^3}{3}$$

$$\int_{(V)} \vec{r}\, dV = \int_{(V)} r\vec{e}_R(\varphi,\vartheta)\, dV = \int_{r=0}^{R}\int_{\vartheta=0}^{\frac{\pi}{2}}\int_{\varphi=0}^{2\pi}\left(r\sin\vartheta\cos\varphi\,\vec{e}_x + r\sin\vartheta\sin\varphi\,\vec{e}_y + r\cos\vartheta\,\vec{e}_z\right)\overbrace{r^2\sin\vartheta\, d\varphi\, d\vartheta\, dr}^{dV}$$

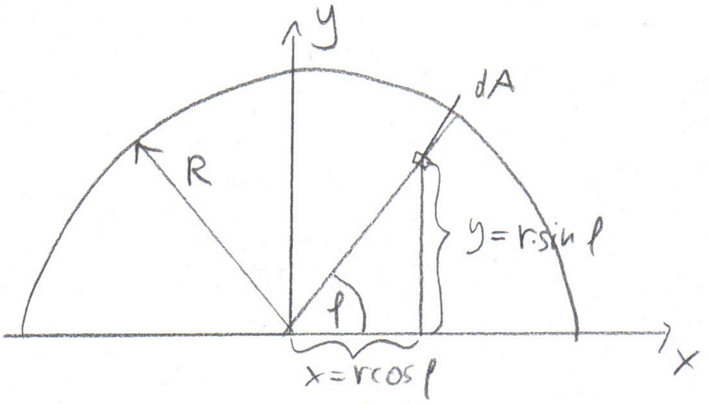
$$=2\pi\int_{r=0}^{R}\int_{\vartheta=0}^{\frac{\pi}{2}} r\cos\vartheta\, r^2\sin\vartheta\, d\vartheta\, dr\,\vec{e}_z = \pi\int_{r=0}^{R}\int_{\vartheta=0}^{\frac{\pi}{2}} r^3\sin 2\vartheta\, d\vartheta\, dr\,\vec{e}_z =$$

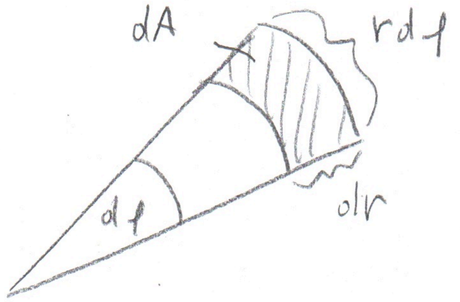
$$=\pi\int_{r=0}^{R} r^3\left[-\frac{\cos 2\vartheta}{2}\right]_0^{\frac{\pi}{2}} dr\,\vec{e}_z = \pi\int_{r=0}^{R} r^3 \underbrace{\left(-\frac{-1}{2}-\left(-\frac{1}{2}\right)\right)}_{1} dr\,\vec{e}_z = \pi\int_{r=0}^{R} r^3\, dr\,\vec{e}_z =$$

$$=\pi\left[\frac{r^4}{4}\right]_0^{R}\vec{e}_z = \frac{R^4\pi}{4}\vec{e}_z$$

$$\vec{r}_S = \frac{\int_{(V)}\vec{r}\, dV}{V} = \frac{\frac{R^4\pi}{4}\vec{e}_z}{\frac{2\pi R^3}{3}} = \frac{R^{\cancel{4}}\cancel{\pi}}{4}\vec{e}_z\,\frac{3}{2\cancel{\pi}\cancel{R^3}} = \boxed{\frac{3}{8}R\,\vec{e}_z}$$

SÍKIDOM SÚLYPONTJA





$$A=\frac{R^2\pi}{2}$$

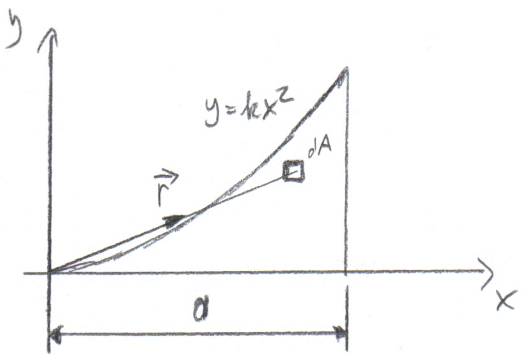
$$\int_{(A)} y\,dA=\int_{\varphi=0}^{\pi}\int_{r=0}^{R}\overbrace{r\cdot\sin\varphi}^{y}\,\overbrace{r\,dr\,d\varphi}^{dA}=\int_{\varphi=0}^{\pi}\int_{r=0}^{R} r^2\sin\varphi\,dr\,d\varphi=$$

$$=\int_{\varphi=0}^{\pi}\left[\frac{r^3}{3}\sin\varphi\right]_0^R d\varphi=\int_{\varphi=0}^{\pi}\frac{R^3}{3}\sin\varphi\,d\varphi=\left[-\frac{R^3}{3}\cos\varphi\right]_{\varphi=0}^{\pi}=$$

$$=\left(-\frac{R^3}{3}\overbrace{\cos\pi}^{-1}\right)-\left(-\frac{R^3}{3}\overbrace{\cos 0^\circ}^{1}\right)=\frac{R^3}{3}+\frac{R^3}{3}=2\frac{R^3}{3}$$

$$\vec{r}_s=\underbrace{x_s\vec{e}_x}_{\vec{0}}+y_s\vec{e}_y=\frac{1}{A}\int_{(A)} y\,dA=\frac{2\frac{R^3}{3}}{\frac{R^2\pi}{2}}=\frac{4R}{3\pi}$$

SÍKIDOM SÚLYPONTJA



$$A=\int_{(A)} dA=\int_{x=0}^{a} kx^2\,dx=\left[k\frac{x^3}{3}\right]_0^a=k\frac{a^3}{3}$$

$$\int_{(A)}\vec{r}\,dA=\int_{(A)}\left(x\vec{e}_x+y\vec{e}_y\right)dA=\int_{(A)}x\,dA\,\vec{e}_x+\int_{(A)}y\,dA\,\vec{e}_y$$

$$\int_{(A)}x\,dA=\int_{x=0}^{a}\int_{y=0}^{kx^2}x\,dy\,dx=\int_{x=0}^{a}\left[xy\right]_{y=0}^{kx^2}dx=\int_{x=0}^{a}kx^3\,dx=$$

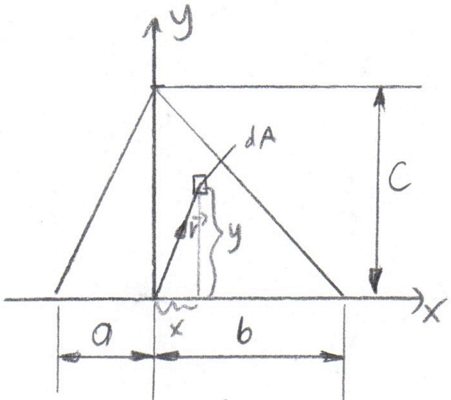
$$=\left[k\frac{x^4}{4}\right]_{x=0}^{a}=k\frac{a^4}{4}$$

$$\int_{(A)}y\,dA=\int_{x=0}^{a}\int_{y=0}^{kx^2}y\,dy\,dx=\int_{x=0}^{a}\left[\frac{y^2}{2}\right]_{y=0}^{kx^2}dx=\int_{x=0}^{a}\frac{k^2x^4}{2}\,dx=$$

$$=\left[\frac{k^2}{2}\frac{x^5}{5}\right]_{x=0}^{a}=\frac{k^2a^5}{10}$$

$$\vec{r}_S=\frac{1}{A}\int_{(A)}x\,dA\,\vec{e}_x+\frac{1}{A}\int_{(A)}y\,dA\,\vec{e}_y=\frac{\cancel{k}\frac{a^4}{4}}{\cancel{k}\frac{a^3}{3}}\vec{e}_x+\frac{k^{\cancel{2}}\frac{a^5}{10}}{\cancel{k}\frac{a^3}{3}}\vec{e}_y=\frac{3}{4}a\,\vec{e}_x+\frac{3}{10}ka^2\,\vec{e}_y$$

SÍKIDOM SÚLYPONTJA



$$A = \int_{(A)} dA = \frac{(a+b)\,c}{2}$$

$$\int_{(A)} \vec{r}\,dA = \int_{(A)} (x\vec{e}_x + y\vec{e}_y)\,dA = \int_{(A)} x\,dA\,\vec{e}_x + \int_{(A)} y\,dA\,\vec{e}_y$$

$$\int_{(A)} x\,dA = \int_{x=-a}^{0}\int_{y=0}^{\frac{c}{a}x+c} x\,dy\,dx + \int_{x=0}^{b}\int_{y=0}^{-\frac{c}{b}x+c} x\,dy\,dx = \int_{x=-a}^{0} x\left(\frac{c}{a}x+c\right)dx + \int_{x=0}^{b} x\left(-\frac{c}{b}x+c\right)dx =$$

$$= \int_{x=-a}^{0} \frac{c}{a}x^2 + cx\,dx + \int_{x=0}^{b} -\frac{c}{b}x^2 + cx\,dx = \left[\frac{c}{a}\frac{x^3}{3} + c\frac{x^2}{2}\right]_{-a}^{0} + \left[-\frac{c}{b}\frac{x^3}{3} + c\frac{x^2}{2}\right]_{0}^{b} =$$

$$= \frac{c\,a^{\cancel{3}\,2}}{\cancel{a}\,3} - \frac{ca^2}{2} - \frac{cb^{\cancel{3}\,2}}{3\cancel{b}} + \frac{cb^2}{2} = \frac{2ca^2}{6} - \frac{3ca^2}{6} - \frac{2cb^2}{6} + \frac{3cb^2}{6} = \frac{c}{6}\left(b^2 - a^2\right)$$

$$\int_{(A)} y\,dA = \int_{x=-a}^{0}\int_{y=0}^{\frac{c}{a}x+c} y\,dy\,dx + \int_{x=0}^{b}\int_{y=0}^{-\frac{c}{b}x+c} y\,dy\,dx = \int_{x=-a}^{0} \frac{1}{2}\left(\frac{c}{a}x+c\right)^2 dx + \int_{x=0}^{b} \frac{1}{2}\left(-\frac{c}{b}x+c\right)^2 dx =$$

$$= \int_{x=-a}^{0} \frac{1}{2}\left(\frac{c^2}{a^2}x^2 + 2\frac{c}{a}xc + c^2\right)dx + \int_{x=0}^{b} \frac{1}{2}\left(\frac{c^2}{b^2}x^2 - 2\frac{c}{b}xc + c^2\right)dx =$$

$$= \frac{1}{2}\left[\frac{c^2x^3}{3a^2} + \frac{\cancel{2}cx^2c}{\cancel{2}a} + c^2x\right]_{-a}^{0} + \frac{1}{2}\left[\frac{c^2x^3}{3b^2} - \frac{\cancel{2}cx^2c}{\cancel{2}b} + c^2x\right]_{0}^{b} =$$

$$= \frac{1}{2}\left(\frac{c^2a^{\cancel{3}}}{3\cancel{a^2}} - \frac{c^2a^{\cancel{2}}}{\cancel{a}} + c^2a + \frac{c^2b^{\cancel{3}}}{3\cancel{b^2}} - \frac{c^2b^{\cancel{2}}}{\cancel{b}} + c^2b\right) =$$

$$= \frac{1}{2}\left(\frac{c^2a}{3} - \cancel{\frac{3c^2a}{3}} + \cancel{\frac{3c^2a}{3}} + \frac{c^2b}{3} - \cancel{\frac{3c^2b}{3}} + \cancel{\frac{3c^2b}{3}}\right) = \frac{c^2}{6}(a+b)$$

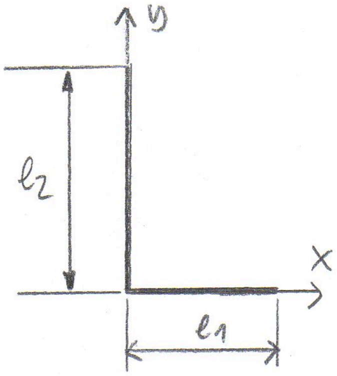
$$\vec{r}_s = \frac{1}{A}\frac{c}{6}(b^2-a^2)\,\vec{e}_x + \frac{1}{A}\frac{c^2}{6}(a+b) =$$

$$= \frac{\cancel{c}\,\overbrace{(b^2-a^2)}^{(\cancel{a+b})(b-a)}}{\cancel{6}_{\,3}} \cdot \frac{\cancel{2}}{\cancel{(a+b)}\cancel{c}}\,\vec{e}_x + \frac{c^{\cancel{2}}\cancel{(a+b)}}{3\cancel{6}} \cdot \frac{\cancel{2}}{\cancel{(a+b)}\cancel{c}}\,\vec{e}_y = \frac{b-a}{3}\,\vec{e}_x + \frac{c}{3}\,\vec{e}_y$$

$$\vec{r_s} = \frac{\cancel{c}\overbrace{(b^2-a^2)}^{\cancel{(a+b)}(b-a)}}{\underset{3}{\cancel{6}}} \cdot \frac{\cancel{2}}{\cancel{(a+b)c}} \vec{e}_x + \frac{c^{\cancel{2}}\cancel{(a+b)}}{3\,\cancel{6}} \cdot \frac{\cancel{2}}{\cancel{(a+b)c}} \vec{e}_y =$$

$$= \frac{b-a}{3} \vec{e}_x + \frac{c}{3} \vec{e}_y$$

VONAL SÚLYPONTJA



$$\ell = \int_{(\ell)} ds = \ell_1 + \ell_2$$

$$\int_{(\ell)} \vec{r}\, ds = \int_{(\ell_2)} y\,\vec{e_y}\, ds + \int_{(\ell_1)} x\,\vec{e_x}\, ds = \int_{y=0}^{\ell_2} y\,dy\,\vec{e_y} + \int_{x=0}^{\ell_1} x\,dx\,\vec{e_x} =$$

$$= \left[\frac{y^2}{2}\right]_{y=0}^{\ell_2}\vec{e_y} + \left[\frac{x^2}{2}\right]_{x=0}^{\ell_1}\vec{e_x} = \frac{\ell_2^2}{2}\vec{e_y} + \frac{\ell_1^2}{2}\vec{e_x} = \frac{1}{2}\left(\ell_1^2\vec{e_x} + \ell_2^2\vec{e_y}\right)$$

$$\vec{r_S} = \frac{1}{\ell}\int_{(\ell_1)} x\,dx\,\vec{e_x} + \frac{1}{\ell}\int_{(\ell_2)} y\,dy\,\vec{e_y} = \frac{1}{2}\left(\frac{\ell_1^2}{\ell_1+\ell_2}\vec{e_x} + \frac{\ell_2^2}{\ell_1+\ell_2}\vec{e_y}\right)$$