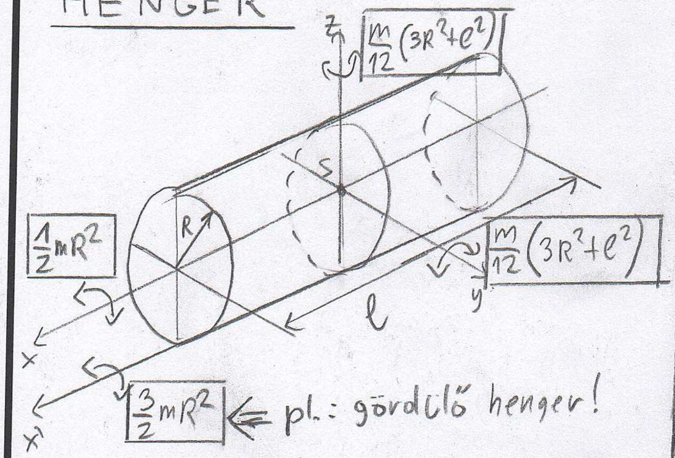
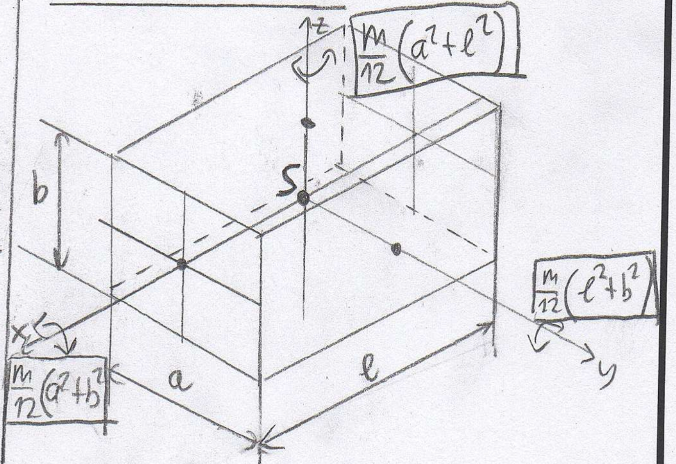
JELLEGZETES MEREV TESTEK TEHETETLENSÉGI NYOMATÉKAI

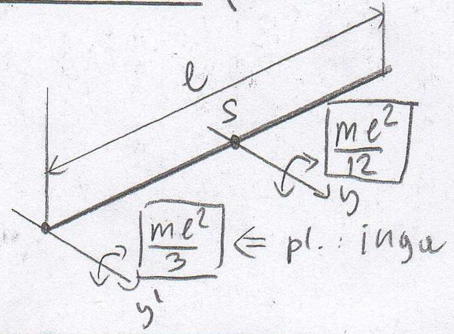
HENGER



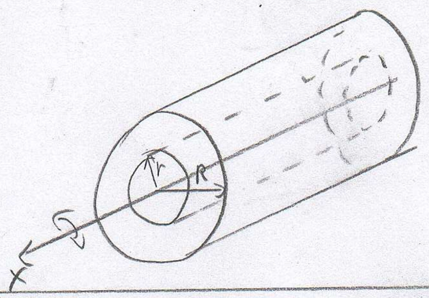
TÉGLATEST



VÉKONY RÚD (fenti kettő határesete, ha R->0 vagy a->0 és b->0)



HENGERGYŰRŰ

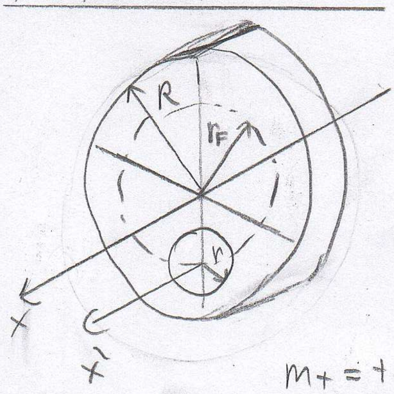


$$J_x = \frac{1}{2} m (R^2 + r^2)$$

mert $J_x = J_x^H - J_x^h = \frac{1}{2}\overbrace{(R^2\pi\ell\rho)}^{m_H}R^2 - \frac{1}{2}\overbrace{(r^2\pi\ell\rho)}^{m_h}r^2 =$

$$= \frac{1}{2}\pi\ell\rho(R^4 - r^4) = \underbrace{\frac{1}{2}\pi\ell\rho(R^2 - r^2)}_{m}(R^2 + r^2)$$

FURATOS TÁRCSA



furatok száma

$$J_x^{FT} = J_x^T - n J_x^F$$

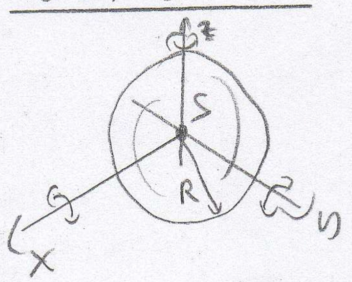
$$J_x^T = \frac{1}{2} m_T R^2$$

$$J_x^F = \underbrace{\frac{1}{2} m_F r^2}_{J_{x'}} + \underbrace{m_F r_F^2}_{\text{Steiner tag}}$$

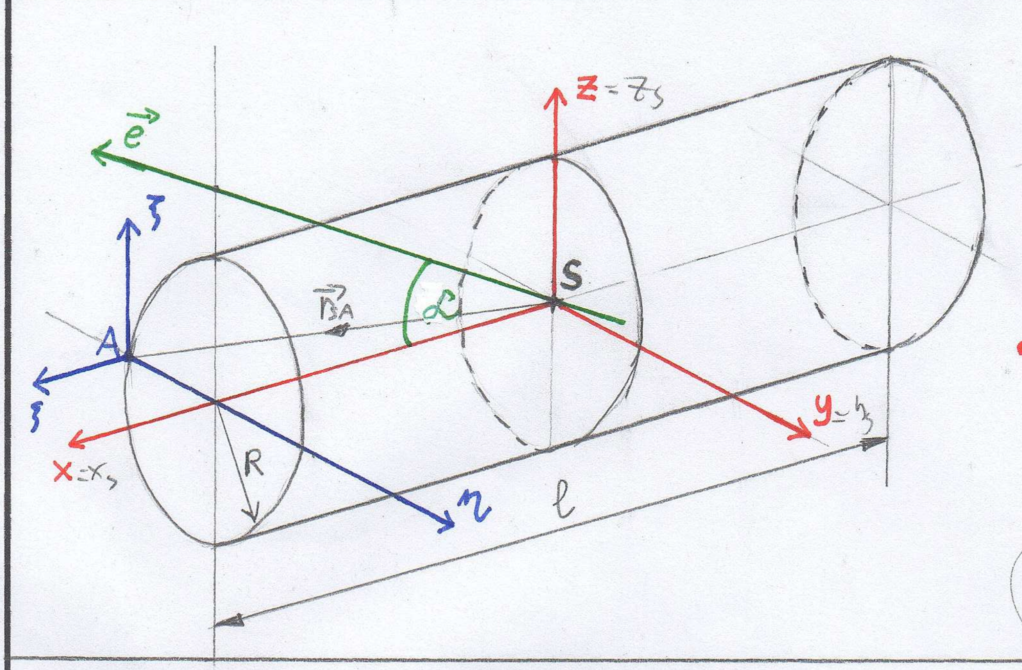
m_T = teli tárcsa tömege

m_F = furat hiányzó tömege

GÖMB



$$J_x = J_y = J_z = \boxed{\frac{2}{5} m R^2}$$



• adott

$m = 10\ kg$

$R = 0{,}4\ m$

$l = 1\ m$

$\alpha = 30^\circ$ (x z sík)

• Feladat

$\underline{\underline{J}}_S = ?$

$J_e = ?$

$\underline{\underline{J}}_A = ?$

$\left([\underline{\underline{J}}_A]_{\xi\eta\zeta} = ? \right)$

• Mivel a vizsgált test tengelyszimmetrikus és

$$\left.\begin{array}{l} x \Rightarrow \text{szimmetriatengely} \\ y \text{ és } z \Rightarrow x\text{-re } \perp, \text{ S ponton átmenő tengelyek} \end{array}\right\} \Rightarrow x, y, z \text{ tehetetlenségi főtengelyek}$$

• A tehetetlenségi főtengelyek által alkotott KR.-ben és így a hozzá tartozó tenzorban is a síkpárra számított tehetetlenségi nyomatékok nullák.

$$\underline{\underline{J}}_S = \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix} = \begin{bmatrix} 0{,}8 & 0 & 0 \\ 0 & 1{,}23 & 0 \\ 0 & 0 & 1{,}23 \end{bmatrix} kg\,m^2$$

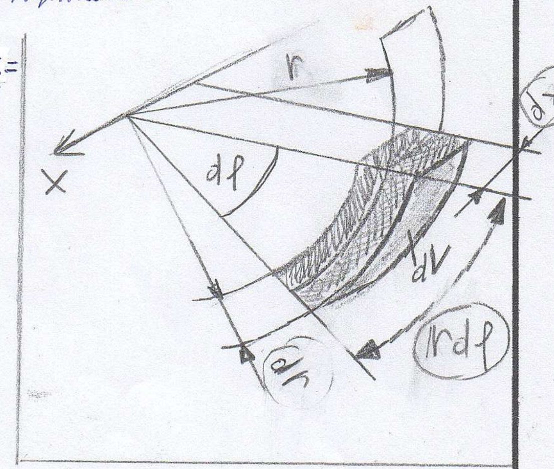
• A szimmetriatengelyre (x) számított tehetetlenségi nyomaték:

$$J_x = \int_{(m)} (y^2+z^2)\,dm = \int_{(V)} \underbrace{(y^2+z^2)}_{r^2} \underbrace{\rho\, dV}_{dm} = \rho \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_0^{2\pi} \int_0^R r^2 \underbrace{dr\, r\, d\varphi\, dx}_{dV} =$$

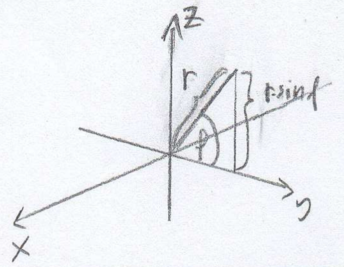
$$= \rho \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^R d\varphi\, dx = \rho \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_0^{2\pi} \frac{R^4}{4}\, d\varphi\, dx =$$

$$= \rho \int_{-\frac{l}{2}}^{\frac{l}{2}} \left[\frac{R^4}{4} \cdot \varphi\right]_0^{2\pi} dx = \rho \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{R^4}{4_2} \cdot 2\pi\, dx =$$

$$= \rho \left[\frac{R^4 \pi}{2} \cdot x\right]_{-\frac{l}{2}}^{\frac{l}{2}} = \rho \frac{R^4}{2} \pi l = \underbrace{\rho \underbrace{R^2 \pi l}_{V}}_{m} \frac{R^2}{2} = \frac{1}{2} m R^2 = \frac{1}{2} \cdot 10 \cdot 0{,}4^2 = 0{,}8\ kg\,m^2$$



• A szimmetriatengelyre merőleges, S ponton átmenő tengelyekre (y és z) számított tehetetlenségi nyomatékok:



$$J_y = \int_{(m)} (x^2+z^2)\,dm = \int_{(V)} (x^2+z^2)\underbrace{\rho\, dV}_{dm} =$$

$$= \rho \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_0^{2\pi} \int_0^R \Big(x^2 + \underbrace{r^2 \sin^2\varphi}_{z^2}\Big) \underbrace{dr\, r\, d\varphi\, dx}_{dV} =$$

$$= \rho \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_0^{2\pi} \int_0^R \left(r x^2 + r^3 \sin^2\varphi\right) dr\, d\varphi\, dx =$$

$$= \rho \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_0^{2\pi} \left[\frac{r^2}{2}x^2 + \frac{r^4}{4}\sin^2\varphi\right]_0^R d\varphi\, dx =$$

$$= \rho \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_0^{2\pi} \frac{R^2}{2}x^2 + \frac{R^4}{4}\cdot \underbrace{\frac{1-\cos 2\varphi}{2}}_{\sin^2\varphi}\, d\varphi\, dx =$$

$$= \rho \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_0^{2\pi} \frac{R^2}{2}x^2 + \frac{R^4}{8} - \frac{R^4}{8}\cdot \cos 2\varphi\, d\varphi\, dx =$$

$$= \rho \int_{-\frac{l}{2}}^{\frac{l}{2}} \left[\frac{R^2}{2}x^2\varphi + \frac{R^4}{8}\varphi - \frac{R^4}{8}\cdot\frac{\sin 2\varphi}{2}\right]_0^{2\pi} dx =$$

$$= \rho \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{R^2}{\cancel{2}}x^2 \cancel{2}\pi + \frac{R^4}{\cancel{8}_4}\cancel{2}\pi\, dx =$$

$$= \rho \left[R^2\pi \frac{x^3}{3} + \frac{R^4\pi}{4}x\right]_{-\frac{l}{2}}^{\frac{l}{2}} =$$

$$= \rho \left[\left(R^2\pi\frac{l^3}{24} + \frac{R^4\pi}{4}\cdot\frac{l}{2}\right) - \left(-R^2\pi\frac{l^3}{24} + \frac{R^4\pi}{4}\cdot\left(-\frac{l}{2}\right)\right)\right] =$$

$$= \cancel{2}\rho R^2\pi \frac{l^3}{\cancel{24}_{12}} + \cancel{2}\rho\frac{R^4\pi}{4}\cdot\frac{l}{\cancel{2}} = \underbrace{\underbrace{R^2\pi l}_{V}\rho}_{m}\left(\frac{l^2}{12} + \frac{R^2}{4}\right) = \frac{m}{12}\left(l^2 + 3R^2\right) =$$

$$= \frac{16}{12}\left(1^2 + 3\cdot 0{,}4^2\right) = 1{,}23\ \text{kg}\,\text{m}^2$$

I $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\alpha = \beta = \varphi$

$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi$

$\hookrightarrow \cos^2\varphi = \cos 2\varphi + \sin^2\varphi$

II $\cos^2\varphi + \sin^2\varphi = 1$

$\hookrightarrow \cos^2\varphi = 1 - \sin^2\varphi$

$1 - \sin^2\varphi = \cos 2\varphi + \sin^2\varphi$

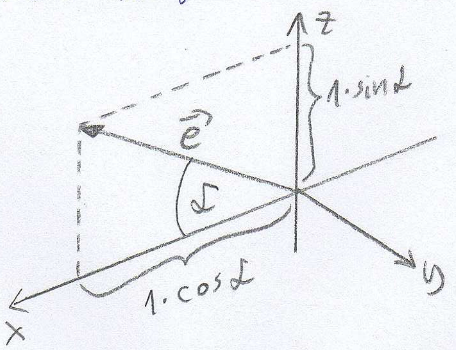
$1 - \cos 2\varphi = 2\sin^2\varphi$

$$\boxed{\sin^2\varphi = \frac{1-\cos 2\varphi}{2}}$$

$J_z = J_y = 1{,}23\ \text{kg}\,\text{m}^2$

• e tengelyre számított tehetetlenségi nyomaték

Az $\vec{e}$ egységvektor xz síkban van



$$\vec{e} = 1\cdot\cos\alpha\,\vec{i} + 0\,\vec{j} + 1\cdot\sin\alpha\,\vec{k}$$

$$\vec{e} = \begin{bmatrix}\cos\alpha & 0 & \sin\alpha\end{bmatrix} \qquad \alpha = 30^\circ$$

$$\vec{e} = \begin{bmatrix}\frac{\sqrt{3}}{2} & 0 & \frac{1}{2}\end{bmatrix}$$

Az S pontra illeszkedő e egyeneshez hozzárendelt tehetetlenségi nyomaték:

$$J_e = \vec{e}\,\underline{\underline{J}}_S\,\vec{e} = \begin{bmatrix}\frac{\sqrt{3}}{2} & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}0{,}8 & 0 & 0\\ 0 & 1{,}23 & 0\\ 0 & 0 & 1{,}23\end{bmatrix}\begin{bmatrix}\frac{\sqrt{3}}{2}\\ 0\\ \frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{3}}{2} & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}0{,}8\cdot\frac{\sqrt{3}}{2}+0\cdot 0+0\cdot\frac{1}{2}\\ 0\cdot\frac{\sqrt{3}}{2}+1{,}23\cdot 0+0\cdot 0\\ 0\cdot 0+0\cdot 0+1{,}23\cdot\frac{1}{2}\end{bmatrix} =$$

$$= \frac{\sqrt{3}}{2}\cdot 0{,}8\,\frac{\sqrt{3}}{2} + 0 + \frac{1}{2}\cdot 1{,}23\cdot\frac{1}{2} = \underline{\underline{0{,}9075\ \mathrm{kgm^2}}}$$

• A pontra vonatkozó tehetetlenségi tenzor

$$x_{SA} = \frac{l}{2} = 0{,}5\,\mathrm{m} \qquad \vec{r}_{SA} = x_{SA}\,\vec{i} + y_{SA}\,\vec{j} + z_{SA}\,\vec{k} =$$

$$y_{SA} = -R = 0{,}4\,\mathrm{m} \qquad = \frac{l}{2}\,\vec{i} - R\,\vec{j} + 0\,\vec{k} = (0{,}5\,\vec{i} - 0{,}4\,\vec{j} + 0\,\vec{k})\,\mathrm{m}$$

$$z_{SA} = 0\,\mathrm{m}$$

Steiner tétel alapján:

$$J_\xi = J_x + m(y_{SA}^2 + z_{SA}^2) = 0{,}8 + 10\left[(-0{,}4)^2 + 0^2\right] = 2{,}4\ \mathrm{kgm^2}$$

$$J_\eta = J_y + m(x_{SA}^2 + z_{SA}^2) = 1{,}23 + 10\cdot\left[0{,}5^2 + 0^2\right] = 3{,}73\ \mathrm{kgm^2}$$

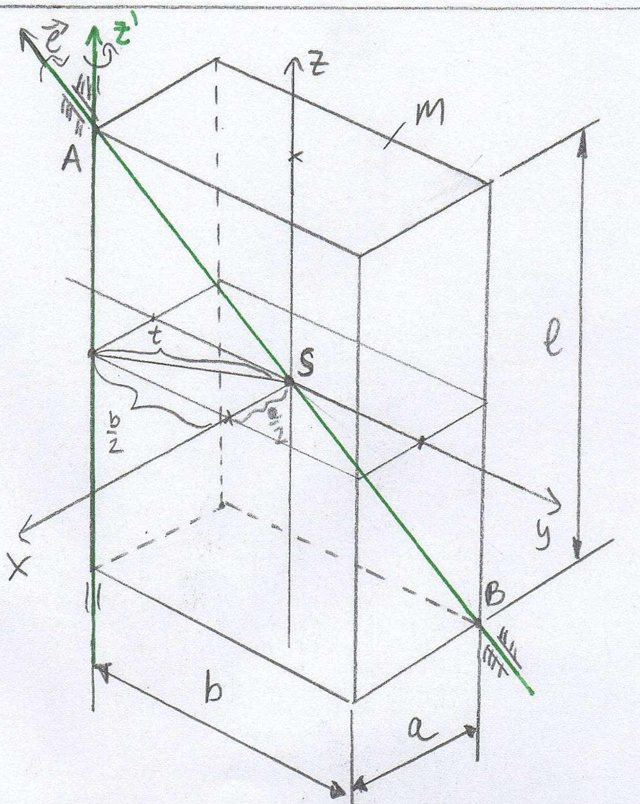
$$J_\zeta = J_z + m(x_{SA}^2 + y_{SA}^2) = 1{,}23 + 10\cdot\left[0{,}5^2 + (-0{,}4)^2\right] = 5{,}33\ \mathrm{kgm^2}$$

$$J_{\xi\eta} = J_{xy} + m\,x_{SA}\,y_{SA} = 0 + 10\cdot 0{,}5\cdot(-0{,}4) = -2\ \mathrm{kgm^2}$$

$$J_{\eta\zeta} = J_{yz} + m\,y_{SA}\,z_{SA} = 0 + 10\cdot(-0{,}4)\cdot 0 = 0\ \mathrm{kgm^2}$$

$$J_{\xi\zeta} = J_{xz} + m\,x_{SA}\,z_{SA} = 0 + 10\cdot 0{,}5\cdot 0 = 0\ \mathrm{kgm^2}$$

$$\underline{\underline{J}}_A = \begin{bmatrix}J_\xi & -J_{\xi\eta} & -J_{\xi\zeta}\\ -J_{\eta\xi} & J_\eta & -J_{\eta\zeta}\\ -J_{\zeta\xi} & -J_{\zeta\eta} & J_\zeta\end{bmatrix} = \begin{bmatrix}2{,}4 & 2 & 0\\ 2 & 3{,}73 & 0\\ 0 & 0 & 5{,}33\end{bmatrix}\ \mathrm{kgm^2}$$



- **adott**

$a = 1\,m$

$b = 2\,m$

$\ell = 3\,m$

$m = 12\,kg$

- **Feladat**

$\underline{\underline{J}}_S = ?$

$J_{z'} = ?$

$J_e = ?$

$$J_x = \int_{(m)} (y^2+z^2)\,dm = \int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}}(y^2+z^2)\,\underbrace{\rho\,\underbrace{dz\,dy\,dx}_{dV}}_{dm} = \rho\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left[y^2 z+\frac{z^3}{3}\right]_{-\frac{\ell}{2}}^{\frac{\ell}{2}}dy\,dx =$$

$$=\rho\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left(\left(y^2\frac{\ell}{2}+\frac{\ell^3}{24}\right)-\left(y^2\left(-\frac{\ell}{2}\right)+\left(-\frac{\ell^3}{24}\right)\right)\right)dy\,dx = \rho\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left(y^2\ell+\frac{\ell^3}{12}\right)dy\,dz =$$

$$=\rho\int_{-\frac{a}{2}}^{\frac{a}{2}}\left[\frac{y^3}{3}\ell+\frac{\ell^3}{12}y\right]_{-\frac{b}{2}}^{\frac{b}{2}}dz = \rho\int_{-\frac{a}{2}}^{\frac{a}{2}}\left(\left(\frac{b^3}{24}\ell+\frac{\ell^3}{12}\cdot\frac{b}{2}\right)-\left(-\frac{b^3}{24}\ell+\frac{\ell^3}{12}\cdot\left(-\frac{b}{2}\right)\right)\right)dz =$$

$$=\rho\int_{-\frac{a}{2}}^{\frac{a}{2}}\left(\frac{b^3}{12}\ell+\frac{\ell^3 b}{12}\right)dz = \rho\left[\frac{\ell b^3}{12}z+\frac{\ell^3 b}{12}z\right]_{-\frac{a}{2}}^{\frac{a}{2}} = \rho\left[\left(\frac{\ell b^3 a}{24}+\frac{\ell^3 b a}{24}\right)-\left(-\frac{\ell b^3 a}{24}-\frac{\ell^3 b a}{24}\right)\right]=$$

$$=\rho\left(\frac{\ell b^3 a}{12}+\frac{\ell^3 b a}{12}\right) = \underbrace{\rho\ell b a}_{m}\frac{b^2}{12}+\underbrace{\rho\ell b a}_{m}\frac{\ell^2}{12} = \frac{m}{12}\left(b^2+\ell^2\right) =$$

$$= \frac{12}{12}\cdot\left(2^2+3^2\right) = \frac{12}{12}\cdot 13 = 13\ kg\,m^2$$

hasonlóan:

$$J_y = \frac{m}{12}\left(a^2+c^2\right) = \frac{12}{12}\left(1^2+3^2\right) = 10\,kg\,m^2$$

$$J_z = \frac{m}{12}\left(a^2+b^2\right) = \frac{12}{12}\cdot\left(1^2+2^2\right) = 5\,kg\,m^2$$

az x y z tehetetlenségi főtengelyek

$$\underline{\underline{J}}_S = \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix} = \begin{bmatrix} 13 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix} kg\,m^2$$

• z' tengelyre számított tehetetlenségi nyomaték

$$J_{z'} = J_z + mt^2 = J_z + m\left[\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2\right] = J_z + m\left(\frac{a^2}{4}+\frac{b^2}{4}\right) =$$

$$= \frac{m}{12}\left(a^2+b^2\right) + \frac{m}{4}\left(a^2+b^2\right) = \frac{m}{3}\left(a^2+b^2\right) = \frac{12}{3}\left(1^2+2^2\right) = \underline{\underline{20\,kg\,m^2}}$$

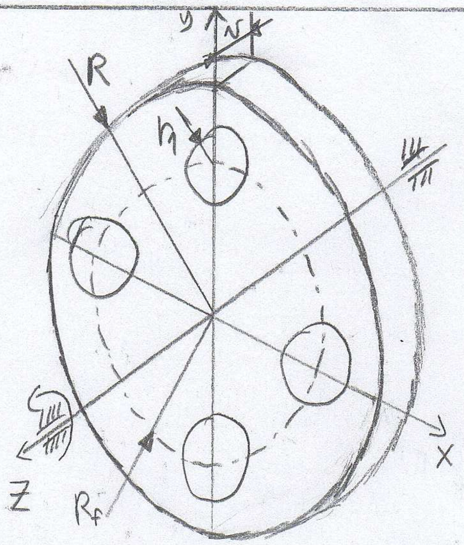
• e tengelyre számított tehetetlenségi nyomaték

$$\vec{e} = \frac{\vec{r}_{BA}}{|\vec{r}_{BA}|} = \frac{a\vec{i} - b\vec{j} + c\vec{k}}{\sqrt{a^2+b^2+c^2}} = \frac{1\vec{i} - 2\vec{j} + 3\vec{k}}{\sqrt{1^2+2^2+3^2}} = \frac{1}{\sqrt{14}}\vec{i} - \frac{2}{\sqrt{14}}\vec{j} + \frac{3}{\sqrt{14}}\vec{k}$$

$$J_e = \vec{e}\cdot\underline{\underline{J}}_S\,\vec{e} = \begin{bmatrix} \frac{1}{\sqrt{14}} & -\frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \end{bmatrix} \begin{bmatrix} 13 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{14}} \\ -\frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{\sqrt{14}} & -\frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \end{bmatrix} \begin{bmatrix} \frac{13}{\sqrt{14}} \\ -\frac{20}{\sqrt{14}} \\ \frac{15}{\sqrt{14}} \end{bmatrix} = \frac{1}{\sqrt{14}}\cdot\frac{13}{\sqrt{14}} + \left(-\frac{2}{\sqrt{14}}\right)\cdot\left(-\frac{20}{\sqrt{14}}\right) + \frac{3}{\sqrt{14}}\cdot\frac{15}{\sqrt{14}} =$$

$$= \frac{13}{14} + \frac{40}{14} + \frac{45}{14} = \frac{98}{14} = \underline{\underline{7\,kg\,m^2}}$$



• adott:

$\rho = 7800 \ \frac{kg}{m^3}$

$v = 0{,}015 \ m$

$R = 0{,}4 \ m$

$R_f = 0{,}2 \ m$

$r_1 = 0{,}1 \ m$

• Feladat

A tárcsa z tengelyre számított tehetetlenségi nyomatéka

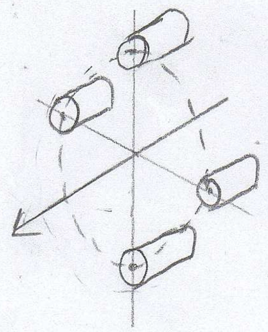
a) furatok nélkül

b) furatokkal

a) $$J_z^{TELI} = \frac{1}{2} m_T R^2 = \frac{1}{2} \cdot 58{,}8 \cdot 0{,}4^2 = \underline{\underline{4{,}704 \ kgm^2}}$$

$$m_T = R^2 \pi v \rho = 0{,}4^2 \pi \, 0{,}015 \cdot 7800 = 58{,}8 \ kg$$

b)



• A 4 darab furat "hiányzó" tehetetlenségi nyomatéka:

$$J_z^{FURAT} = \frac{1}{2} m_1 r_1^2 + m_1 R_f^2 \quad (1 \ db)$$

$$m_1 = r_1^2 \pi v \cdot \rho = 0{,}1^2 \cdot \pi \cdot 0{,}015 \cdot 7800 =$$
$$= 3{,}6756 \ kg \quad (\text{tömeghiány})$$

$$J_z^{FURATOK} = 4 \cdot \left[\frac{1}{2} m_1 r_1^2 + m_1 R_f^2\right] =$$
$$= 4 \cdot \left[\frac{1}{2} \cdot 3{,}6756 \cdot 0{,}1^2 + 3{,}6756 \cdot 0{,}2^2\right] =$$
$$= 0{,}6616 \ kgm^2$$

$$J_z^{FURATOS \ TÁRCSA} = J_z^{TELI} - J_z^{FURATOK} = 4{,}704 - 0{,}6616 = \underline{\underline{4{,}0424 \ kgm^2}}$$