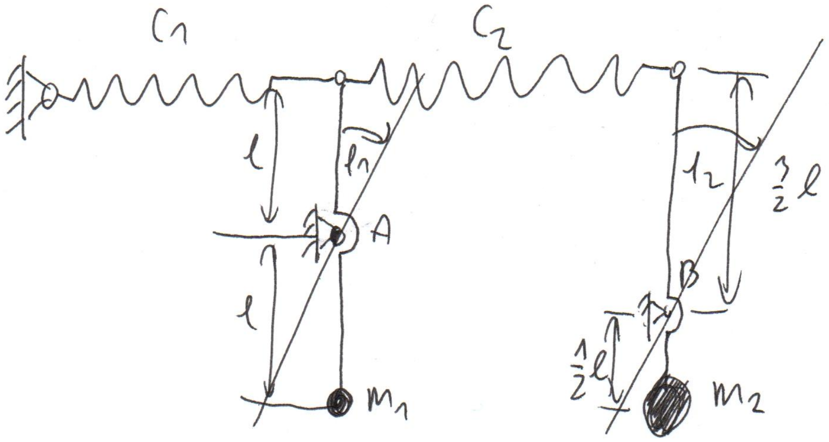


adott:

$m_1 = 60\,kg$

$m_2 = 120\,kg$

$l = 1\,m$

$c_1 = c_2 = 4\cdot 10^{-4}\,\frac{m}{N}$

ált. koord: $\varphi_1\ \varphi_2$

– DOFok száma: 2

– ált. koord: $\underline{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$ $\quad \dot{\underline{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot\varphi_1 \\ \dot\varphi_2 \end{bmatrix}$ $\quad \ddot{\underline{q}} = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} \ddot\varphi_1 \\ \ddot\varphi_2 \end{bmatrix}$

Lagrange egyenlet: $\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_i}\right) - \frac{\partial E}{\partial q_i} = Q_{ci} \qquad i = 1,2$

· A rendszer teljes kinetikai energiája:

$$E = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 (l\dot\varphi_1)^2 + \frac{1}{2} m_2 \left(\frac{1}{2} l \dot\varphi_2\right)^2 =$$

$$= \frac{1}{2} m_1 l^2 \dot\varphi_1^2 + \frac{1}{2} m_2 \frac{1}{4} l^2 \dot\varphi_2^2$$

· A rendszer teljes rugóenergiája:

$$V = V_1 + V_2 = \frac{1}{2}\frac{f_1^2}{c_1} + \frac{1}{2}\frac{f_2^2}{c_2} = \frac{1}{2}\frac{(l\varphi_1)^2}{c_1} + \frac{1}{2}\frac{\left(\frac{3}{2} l \varphi_2 - l\varphi_1\right)^2}{c_2}$$

Az 1. mozgásegyenlet $\boxed{i=1}$

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot\varphi_1}\right)-\underbrace{\frac{\partial E}{\partial \varphi}}_{=0}=Q_{c1}$$

- $$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot\varphi_1}\right)=m_1 l^2 \ddot\varphi_1$$

- $$Q_{c1}=-\frac{\partial U}{\partial \varphi_1}=-\left(\frac{l^2\varphi_1}{c_1}+\frac{\frac{3}{2}l\varphi_2-l\varphi_1}{c_2}\cdot(-l)\right)=$$

$$=-\frac{l^2\varphi_1}{c_1}+\frac{\frac{3}{2}l^2\varphi_2-l\varphi_1}{c_2}=-\frac{l^2}{c_1}\varphi_1+\frac{3l^2}{2c_2}\varphi_2-\frac{l}{c_2}\varphi_1=$$

$$=-\left(\frac{l^2}{c_1}+\frac{l^2}{c_2}\right)\varphi_1+\frac{3l^2}{2c_2}\varphi_2$$

A 2. mozgásegyenlet: $\boxed{i=2}$

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot\varphi_2}\right)-\frac{\partial E}{\partial \varphi}=Q_{c2}$$

- $$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot\varphi_2}\right)=\frac{1}{4}m_2 l^2 \ddot\varphi_2$$

- $$Q_{c2}=-\frac{\partial U}{\partial \varphi_2}=-\left(\frac{\frac{3}{2}l\varphi_2-l\varphi_1}{c_2}\cdot\frac{3}{2}l\right)=$$

$$=-\frac{\frac{9}{4}l^2\varphi_2-\frac{3}{2}l^2\varphi_1}{c_2}=-\frac{9l^2}{4c_2}\varphi_2+\frac{3l^2}{2c_2}\varphi_1=\frac{3l^2}{2c_2}\varphi_1-\frac{9l^2}{4c_2}\varphi_2$$

visszahelyettesítve a Lagrange egyenletbe a mozgásegyenlet rendszer:

$$\left.\begin{aligned} \boxed{i=1} &\Rightarrow m_1 l^2 \ddot{\varphi}_1 + \left(\frac{l^2}{c_1}+\frac{l^2}{c_2}\right)\varphi_1 - \frac{3l^2}{2c_2}\varphi_2 = 0 \\ \boxed{i=2} &\Rightarrow \frac{1}{4} m_2 l^2 \ddot{\varphi}_2 - \frac{3l^2}{2c_2}\varphi_1 + \frac{9l^2}{4c_2}\varphi_2 = 0 \end{aligned}\right\}$$

Mátrixos alakban felírva:

$$\underbrace{\begin{bmatrix} m_1 l^2 & 0 \\ 0 & \frac{1}{4} m_2 l^2 \end{bmatrix}}_{\underline{\underline{M}}} \underbrace{\begin{bmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{bmatrix}}_{\underline{\underline{\ddot{q}}}} + \underbrace{\left[\begin{array}{c|c} \frac{l^2}{c_1}+\frac{l^2}{c_2} & -\frac{3l^2}{2c_1} \\ \hline -\frac{3l^2}{2c_1} & \frac{9l^2}{4c_2} \end{array}\right]}_{\underline{\underline{K}}} \underbrace{\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}}_{\underline{\underline{q}}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{\underline{M}}\,\underline{\underline{\ddot{q}}} + \underline{\underline{K}}\,\underline{\underline{q}} = 0$$

a tömegmátrix: $\underline{\underline{M}} = \begin{bmatrix} m_1 l^2 & 0 \\ 0 & \frac{1}{2} m_2 l^2 \end{bmatrix} = \begin{bmatrix} 60\cdot 1^2 & 0 \\ 0 & \frac{1}{4} 120\cdot 1^2 \end{bmatrix} = \begin{bmatrix} 60 & 0 \\ 0 & 30 \end{bmatrix} \text{kgm}^2$

arugómátrix:
(merevségimátrix)

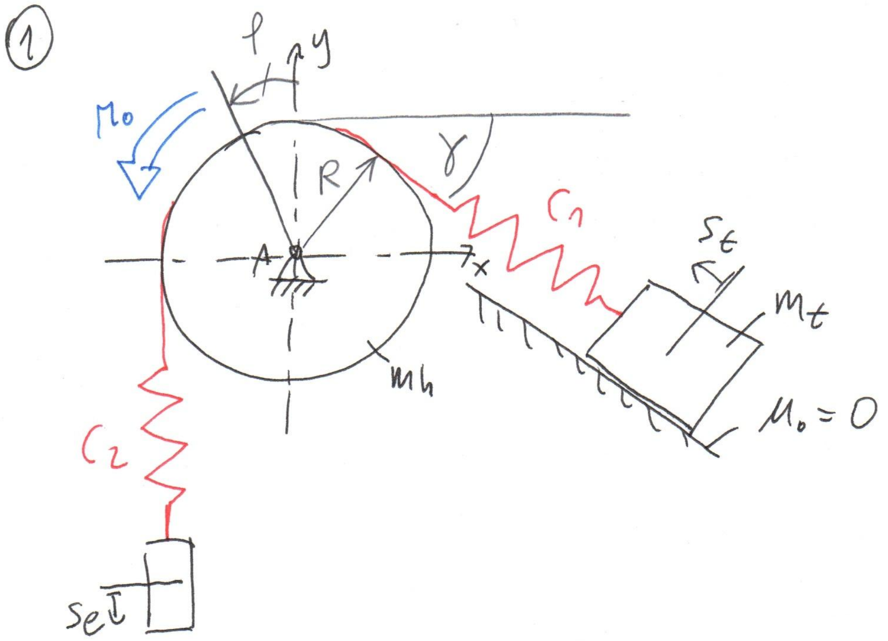
$$\underline{\underline{K}} = \left[\begin{array}{c|c} \frac{l^2}{c_1}+\frac{l^2}{c_2} & -\frac{3l^2}{2c_1} \\ \hline -\frac{3l^2}{2c_1} & \frac{9l^2}{4c_2} \end{array}\right] = \left[\begin{array}{c|c} \frac{1^2}{4\cdot 10^{-4}}+\frac{1^2}{4\cdot 10^{-4}} & -\frac{3\cdot 1^2}{2\cdot 4\cdot 10^{-4}} \\ \hline -\frac{3\cdot 1^2}{2\cdot 4\cdot 10^{-4}} & \frac{9\cdot 1^2}{4\cdot 10^{-4}} \end{array}\right] =$$

$$= \begin{bmatrix} 5000 & -3750 \\ -3750 & 5625 \end{bmatrix} \text{N}\cdot\text{m}$$

A mozgásegyenlet:

$$\begin{bmatrix} 60 & 0 \\ 0 & 30 \end{bmatrix} \begin{bmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{bmatrix} + \begin{bmatrix} 5000 & -3750 \\ -3750 & 5625 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

①



adott:

$m_h, m_e, m_t,$

c_1, c_2, R, γ

$M_0 = 0$

$M_0 =$ áll.

ált. koordináták:

s_t, φ, s_e

általános koordináták:

$q_1 = s_t \quad q_2 = \varphi \quad q_3 = s_e$

$\dot{q}_1 = \dot{s}_t \quad \dot{q}_2 = \dot{\varphi} \quad \dot{q}_3 = \dot{s}_e$

$\ddot{q}_1 = \ddot{s}_t \quad \ddot{q}_2 = \ddot{\varphi} \quad \ddot{q}_3 = \ddot{s}_e$

$$\underline{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} s_t \\ \varphi \\ s_e \end{bmatrix} \quad \underline{\dot{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} \dot{s}_t \\ \dot{\varphi} \\ \dot{s}_e \end{bmatrix} \quad \underline{\ddot{q}} = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} = \begin{bmatrix} \ddot{s}_t \\ \ddot{\varphi} \\ \ddot{s}_e \end{bmatrix}$$

Lagrange egyenlet: $\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_i}\right) - \frac{\partial E}{\partial q_i} = Q_{ci} + Q_{gi} \quad i = 1,2,3$

---

① • A rendszer teljes kinetikai energiája:

$$E = \frac{1}{2} m_t v_t^2 + \frac{1}{2} J_a \omega^2 + \frac{1}{2} m_e v_e^2 =$$

$$= \frac{1}{2} m_t \dot{s}_t^2 + \frac{1}{2}\left(\frac{1}{2} m_h R^2\right) \dot{\varphi}^2 + \frac{1}{2} m_e \dot{s}_e^2 = \left(\frac{1}{2} m_t \dot{q}_1^2 + \frac{1}{2}\left(\frac{1}{2} m_h R^2\right)\dot{q}_2^2 + \frac{1}{2} m_e \dot{q}_3^2\right)$$

② • A rendszer teljes rugóenergiája:

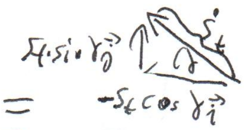
$$U = U_1 + U_2 = \frac{1}{2}\frac{f_1^2}{c_1} + \frac{1}{2}\frac{f_2^2}{c_2} = \frac{1}{2}\frac{(R\varphi - s_t)^2}{c_1} + \frac{1}{2}\frac{(s_e - R\varphi)^2}{c_2} =$$

$$\left(= \frac{1}{2}\frac{(Rq_2 - q_1)^2}{c_1} + \frac{1}{2}\frac{(q_3 - Rq_2)^2}{c_2}\right)$$

• A rendszerre ható külső erőrendszer:

$$\vec M_0;\quad \vec G_t = m_t\vec g;\quad \underset{\hookrightarrow\, P=0}{\vec G_h = m_h\vec g},\quad \vec G_e = m_e\vec g,\quad \underbrace{\overbrace{\vec F_t,\ \vec F_A}^{\text{kényszererők}}}_{P=0}$$

③ A külső ER teljesítménye:



$$P = \vec M_0\vec\omega + \vec G_t\cdot\vec v_t + \vec G_e\vec v_e =$$

$$= M_0\vec k\cdot\dot\varphi\vec k - m_t g\vec j\cdot\overbrace{\dot s_t\left(-\cos\gamma\,\vec i + \sin\gamma\,\vec j\right)}^{\vec v_t} - m_e g\vec j\cdot\overbrace{\left(-\dot s_e\vec j\right)}^{\vec v_e} =$$

$$= M_0\dot\varphi - m_t g\dot s_t\sin\gamma + m_e g\dot s_e$$

Az 1. mozgásegyenlet $\boxed{i=1}$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial E}{\partial\dot s_t}\right)}_{①} - \underbrace{\frac{\partial E}{\partial s_t}}_{=0} = \underbrace{Q_{c1}}_{②} + \underbrace{Q_{g1}}_{③}$$

① $\frac{d}{dt}\left(\frac{\partial E}{\partial\dot s_t}\right) = 2\cdot\frac{1}{2}m_t\ddot s_t = m_t\ddot s_t$

② $Q_{c1} = -\frac{\partial U}{\partial q_1} = -\frac{\partial U}{\partial s_t} = -\frac{R\varphi - s_t}{c_1}\cdot(-1) = \frac{+R\varphi - s_t}{c_1} = -\frac{1}{c_1}s_t + \frac{R}{c_1}\varphi$

③ $Q_{g1} = \frac{\partial P}{\partial\dot q_1} = \frac{\partial P}{\partial\dot s_t} = -m_t g\sin\gamma$

A 2. mozgásegyenlet: $\boxed{i=2}$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial E}{\partial\dot\varphi}\right)}_{①} - \underbrace{\frac{\partial E}{\partial\varphi}}_{=0} = \underbrace{Q_{c2}}_{②} + \underbrace{Q_{g2}}_{③}$$

① $\frac{d}{dt}\left(\frac{\partial E}{\partial\dot\varphi}\right) = \frac{1}{2}m_h R^2\ddot\varphi$

② $$Q_{c2} = -\frac{\partial U}{\partial q_2} = -\frac{\partial U}{\partial \varphi} = -\left(\frac{R\varphi - s_t}{c_1} \cdot R + \frac{s_e - R\varphi}{c_2} \cdot (-R)\right) =$$

$$= -\frac{R^2\varphi}{c_1} + \frac{s_t R}{c_1} + \frac{R s_e}{c_2} - \frac{R^2\varphi}{c_2} = +\frac{R}{c_1} s_t - \left(\frac{R^2}{c_1} + \frac{R^2}{c_2}\right)\varphi + \frac{R}{c_2} s_e$$

③ $$Q_{y2} = \frac{\partial P}{\partial \dot{q}_2} = \frac{\partial P}{\partial \dot{\varphi}} = M_0$$

A 3. mozgásegyenlet $\boxed{i=3}$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{s}_e}\right)}_{①} - \underbrace{\frac{\partial E}{\partial s_e}}_{=0} = \underbrace{Q_{c3}}_{②} + \underbrace{Q_{y3}}_{③}$$

① $$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{s}_e}\right) = m_e \ddot{s}_e$$

② $$Q_{c3} = -\frac{\partial U}{\partial q_3} = -\frac{\partial U}{\partial s_e} = -\frac{s_e - R\varphi}{c_2} = +\frac{R}{c_2}\varphi - \frac{1}{c_2} s_e$$

③ $$Q_{y3} = \frac{\partial P}{\partial \dot{q}_3} = \frac{\partial P}{\partial \dot{s}_e} = m_e g$$

---

visszahelyettesítve a mozgásegyenletrendszer:

$$\left.\begin{aligned}
\boxed{i=1} &\rightarrow m_t \ddot{s}_t + \frac{1}{c_1} s_t - \frac{R}{c_1}\varphi = -m_t g \sin\gamma \\
\boxed{i=2} &\rightarrow \frac{1}{2} m_h R^2 \ddot{\varphi} - \frac{R}{c_1} s_t + \left(\frac{R^2}{c_1} + \frac{R^2}{c_2}\right)\varphi - \frac{R}{c_2} s_e = M_0 \\
\boxed{i=3} &\rightarrow m_e \ddot{s}_e - \frac{R}{c_2}\varphi + \frac{1}{c_2} s_e = m_e g
\end{aligned}\right\}$$

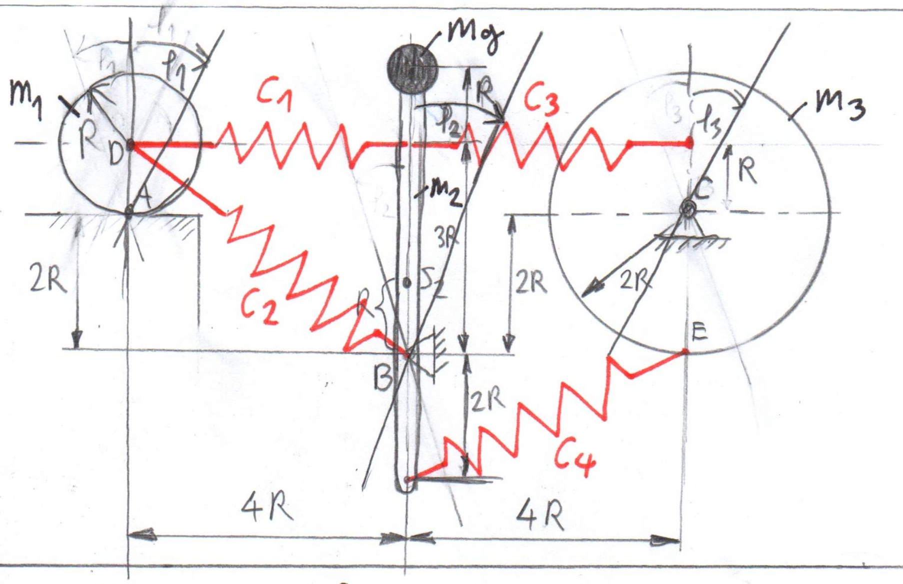
Felírás mátrixos alakban

$$
\begin{array}{l} \\ 1.\text{egy}\to \\ 2.\text{egy}\to \\ 3.\text{egy}\to \end{array}
\underbrace{\left[\begin{array}{ccc} \overset{\ddot{s}_t}{\downarrow} & \overset{\ddot{\varphi}}{\downarrow} & \overset{\ddot{s}_e}{\downarrow} \\ m_t & 0 & 0 \\ 0 & \frac{1}{2} m_h R^2 & 0 \\ 0 & 0 & m_e \end{array}\right]}_{\underline{\underline{M}}}
\underbrace{\begin{bmatrix} \ddot{s}_t \\ \ddot{\varphi} \\ \ddot{s}_e \end{bmatrix}}_{\underline{\ddot{q}}}
+
\underbrace{\left[\begin{array}{c|c|c} \overset{s_t}{\downarrow} & \overset{\varphi}{\downarrow} & \overset{s_e}{\downarrow} \\ \frac{1}{c_1} & -\frac{R}{c_1} & 0 \\ \hline -\frac{R}{c_1} & \frac{R^2}{c_1} + \frac{R^2}{c_2} & -\frac{R}{c_2} \\ \hline 0 & -\frac{R}{c_2} & \frac{1}{c_2} \end{array}\right]}_{\underline{\underline{K}}}
\underbrace{\begin{bmatrix} s_t \\ \varphi \\ s_e \end{bmatrix}}_{\underline{q}}
=
\underbrace{\begin{bmatrix} -m_t g \sin\gamma \\ M_0 \\ m_e g \end{bmatrix}}_{\underline{\underline{Q_g}}}
$$

$\underline{\underline{M}}$ ↳ tömegmátrix

$\underline{\underline{K}}$ ↳ rugómátrix (merevségimátrix)

$$\underline{\underline{M}}\,\underline{\ddot{q}} + \underline{\underline{K}}\,\underline{q} = \underline{\underline{Q_g}}$$



adott:

m_1 c_1 R

m_2 c_2

m_3 c_3

m_g c_4

ált. koord.:

$\varphi_1, \varphi_2, \varphi_3$

DOFok száma: 3

általános koordináták: $\underline{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix}$; $\dot{\underline{q}} = \begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \dot q_3 \end{bmatrix} = \begin{bmatrix} \dot\varphi_1 \\ \dot\varphi_2 \\ \dot\varphi_3 \end{bmatrix}$, $\ddot{q} = \begin{bmatrix} \ddot q_1 \\ \ddot q_2 \\ \ddot q_3 \end{bmatrix} = \begin{bmatrix} \ddot\varphi_1 \\ \ddot\varphi_2 \\ \ddot\varphi_3 \end{bmatrix}$

Lagrange egyenlet: $\frac{d}{dt}\left(\frac{\partial E}{\partial \dot q_i}\right) - \frac{\partial E}{\partial q_i} = Q_{ci}$ $\quad i = 1, 2, 3$

A rendszer teljes kinetikai energiája:

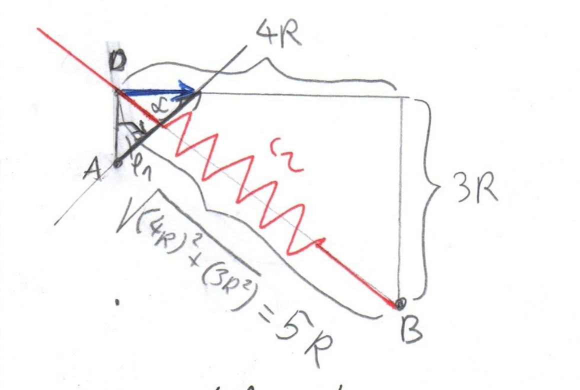
$$E = \frac{1}{2} J_a \omega_1^2 + \frac{1}{2} J_b \omega_2^2 + \frac{1}{2} m_g v_g^2 + \frac{1}{2} J_c \omega_3^2 =$$

$$= \frac{1}{2}\left(\frac{3}{2} m_1 R^2\right) \dot\varphi_1^2 + \frac{1}{2}\left(\frac{1}{12} m_2 (6R)^2 + m_2 R^2\right) \dot\varphi_2^2 + \frac{1}{2} m_g \left(4R\dot\varphi_2\right)^2 +$$

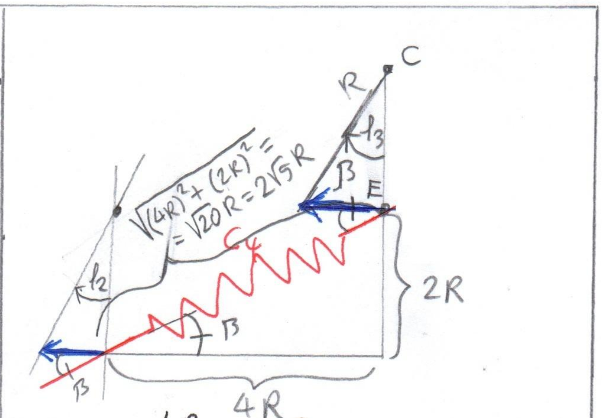
$$+ \frac{1}{2}\left(\frac{1}{2} m_3 (2R)^2\right) \dot\varphi_3^2 =$$

$$= \frac{1}{2}\left(\frac{3}{2} m_1 R^2\right) \dot\varphi_1^2 + \frac{1}{2}\left(4 m_2 R^2\right) \dot\varphi_2^2 + \frac{1}{2}\left(m_g 16 R^2\right) \dot\varphi_2^2 + \frac{1}{2}\left(2 m_3 R^2\right) \dot\varphi_3^2$$

A rendszer teljes rugóenergiája



$$\cos\alpha = \frac{4R}{5R} = \frac{4}{5}$$



$$\cos\beta = \frac{4R}{2\sqrt{5}R} = \frac{2}{\sqrt{5}}$$

$$U = U_1+U_2+U_3+U_4 = \frac{1}{2}\frac{f_1^2}{c_1} + \frac{1}{2}\frac{f_2^2}{c_2} + \frac{1}{2}\frac{f_3^2}{c_3} + \frac{1}{2}\frac{f_4^2}{c_4} =$$

$$= \frac{1}{2}\frac{(3R\varphi_2 - R\varphi_1)^2}{c_1} + \frac{1}{2}\frac{(R\varphi_1 \overset{\cos\alpha}{\frac{4}{5}})^2}{c_2} + \frac{1}{2}\frac{(R\varphi_3 - 3R\varphi_2)^2}{c_3} + \frac{1}{2}\frac{\left(2R\varphi_2 \overset{\cos\beta}{\frac{2}{\sqrt{5}}} - 2R\varphi_3 \overset{\cos\beta}{\frac{2}{\sqrt{5}}}\right)^2}{c_4}$$

- Az 1. mozgásegyenlet: $\boxed{i=1}$

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\varphi}_1}\right) - \underbrace{\frac{\partial E}{\partial \varphi_1}}_{=0} = Q_{c_1}$$

- $$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\varphi}_1}\right) = \frac{3}{2} m_1 R^2 \ddot{\varphi}_1$$

- $$Q_{c_1} = -\frac{\partial U}{\partial \varphi_1} = -\left(\frac{3R\varphi_2 - R\varphi_1}{c_1}\cdot(-R) + \frac{R\varphi_1\frac{4}{5}}{c_2}\cdot R\frac{4}{5}\right) =$$

$$= \frac{3R^2\varphi_2 - R^2\varphi_1}{c_1} - \frac{\frac{16}{25}R^2\varphi_1}{c_2} = \left(-\frac{R^2}{c_1} - \frac{16R^2}{25c_2}\right)\varphi_1 + \frac{3R^2}{c_1}\varphi_2 =$$

$$= -\left(\frac{R^2}{c_1} + \frac{16R^2}{25c_2}\right)\varphi_1 + \frac{3R^2}{c_1}\varphi_2$$

A 2. mozgásegyenlet $\boxed{i=2}$

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\varphi}_2}\right)-\frac{\partial E}{\partial \varphi_2}=Q_{c2}$$

$$\bullet\ \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\varphi}_2}\right)=4M_2R^2\ddot{\varphi}_2+M_4 16R^2\ddot{\varphi}_2=\left(4m_2R^2+16m_4R^2\right)\ddot{\varphi}_2$$

$$\bullet\ Q_{c2}=-\frac{\partial U}{\partial \varphi_2}=-\left(\frac{3R\varphi_2-R\varphi_1}{c_1}(3R)+\frac{R\varphi_3-3R\varphi_2}{c_3}(-3R)+\right.$$

$$\left.+\frac{2R\varphi_2\frac{2}{\sqrt{5}}-2R\varphi_3\frac{2}{\sqrt{5}}}{c_4}\cdot\left(2R\frac{2}{\sqrt{5}}\right)\right)=$$

$$=\frac{-9R^2\varphi_2+3R^2\varphi_1}{c_1}+\frac{3R^2\varphi_3-9R^2\varphi_2}{c_3}-\frac{\frac{16}{5}R^2\varphi_2-\frac{16}{5}R^2\varphi_3}{c_4}=$$

$$=-\frac{9R^2}{c_1}\varphi_2+\frac{3R^2}{c_1}\varphi_1+\frac{3R^2}{c_3}\varphi_3-\frac{9R^2}{c_3}\varphi_2-\frac{16R^2}{5c_4}\varphi_2+\frac{16R^2}{5c_4}\varphi_3=$$

$$=\frac{3R^2}{c_1}\varphi_1-\left(\frac{9R^2}{c_3}+\frac{9R^2}{c_1}+\frac{16R^2}{5c_4}\right)\varphi_2+\left(\frac{16R^2}{5c_4}+\frac{3R^2}{c_3}\right)\varphi_3$$

A 3. mozgásegyenlet $\boxed{i=3}$

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\varphi}_3}\right)-\frac{\partial E}{\partial \varphi_3}=Q_{c3}$$

$$\bullet\ \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\varphi}_3}\right)=2m_3R^2\ddot{\varphi}_3$$

$$\bullet\ Q_{c3}=-\frac{\partial U}{\partial \varphi_3}=-\left(\frac{R\varphi_3-3R\varphi_2}{c_3}\cdot R+\frac{2R\varphi_2\frac{2}{\sqrt{5}}-2R\varphi_3\frac{2}{\sqrt{5}}}{c_4}\cdot\left(-2R\frac{2}{\sqrt{5}}\right)\right)=$$

$$=-\frac{R^2\varphi_3-3R^2\varphi_2}{c_3}+\frac{\frac{16}{5}R^2\varphi_2-\frac{16}{5}R^2\varphi_3}{c_4}=$$

$$=-\frac{R^2}{c_3}\varphi_3+\frac{3R^2}{c_3}\varphi_2+\frac{16R^2}{5c_4}\varphi_2-\frac{16R^2}{5c_4}\varphi_3=$$

$$=\left(\frac{3R^2}{c_3}+\frac{16R^2}{5c_4}\right)\varphi_2-\left(\frac{R^2}{c_3}+\frac{16R^2}{5c_4}\right)\varphi_3$$

· visszahelyettesítve a Lagrange egyenletekbe:

$$\boxed{i=1} \Rightarrow \frac{3}{2} m_1 R^2 \ddot{\varphi}_1 + \left(\frac{R^2}{c_1} + \frac{16R^2}{25c_2}\right)\varphi_1 - \frac{3R^2}{c_1}\varphi_2 = 0$$

$$\boxed{i=2} \Rightarrow \left(4m_2R^2 + 16m_4R^2\right)\ddot{\varphi}_2 - \frac{3R^2}{c_1}\varphi_1 + \left(\frac{9R^2}{c_3} + \frac{9R^2}{c_1} + \frac{16R^2}{5c_4}\right)\varphi_2 - \left(\frac{16R^2}{5c_4} + \frac{3R^2}{c_3}\right)\varphi_3 = 0$$

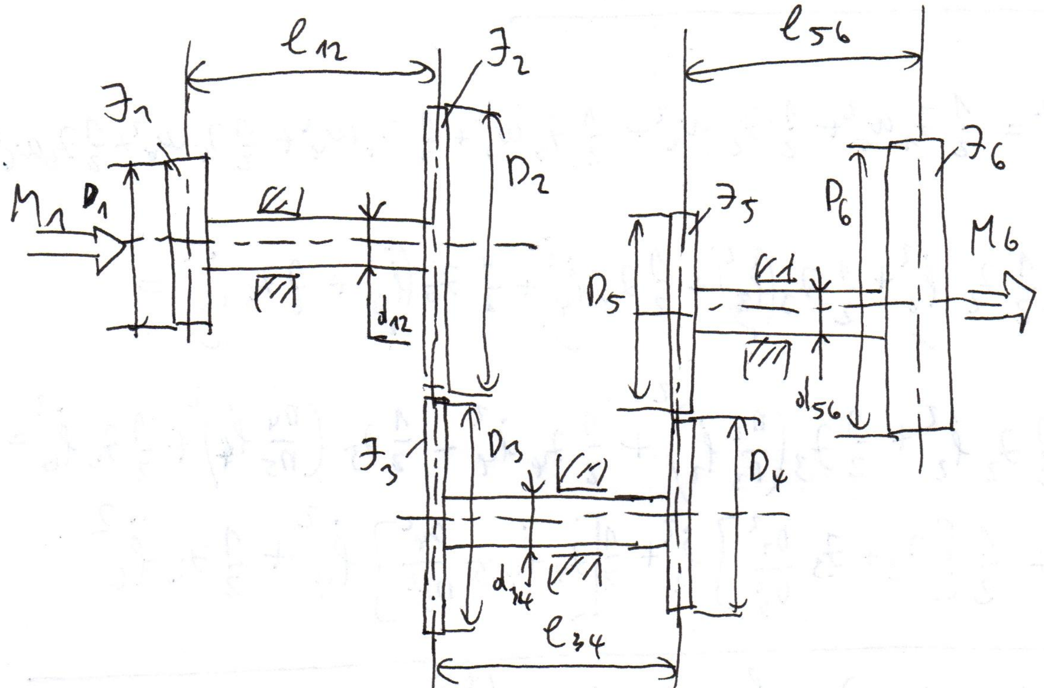
$$\boxed{i=3} \Rightarrow 2m_3R^2\ddot{\varphi}_3 - \left(\frac{3R^2}{c_3} + \frac{16R^2}{5c_4}\right)\varphi_2 + \left(\frac{R^2}{c_3} + \frac{16R^2}{5c_4}\right)\varphi_3 = 0$$

· Felírás mátrixos alakban:

$$\underbrace{\begin{bmatrix} \frac{3}{2}m_1R^2 & 0 & 0 \\ 0 & 4m_2R^2+16m_4R^2 & 0 \\ 0 & 0 & 2m_3R^2 \end{bmatrix}}_{\underline{\underline{M}}} \underbrace{\begin{bmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \\ \ddot{\varphi}_3 \end{bmatrix}}_{\underline{\ddot{q}}} +$$

$$+ \underbrace{\begin{bmatrix} \frac{R^2}{c_1} + \frac{16R^2}{25c_2} & -\frac{3R^2}{c_1} & 0 \\ -\frac{3R^2}{c_1} & \frac{9R^2}{c_3} + \frac{9R^2}{c_2} + \frac{16R^2}{5c_4} & -\left(\frac{16R^2}{5c_4} + \frac{3R^2}{c_3}\right) \\ 0 & -\left(\frac{3R^2}{c_3} + \frac{16R^2}{5c_4}\right) & \frac{R^2}{c_3} + \frac{16R^2}{5c_4} \end{bmatrix}}_{\underline{\underline{K}}} \underbrace{\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix}}_{\underline{q}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{\underline{Q_q}}}$$

HAJTÓMŰ TORZIÓS REZGÉSEI



Adott:

$J_1\ J_2\ J_3\ J_4\ J_5\ J_6$

$D_1\ D_2\ D_3\ D_4\ D_5\ D_6$

$l_{12}\ l_{34}\ l_{56}$

$d_{12}\ d_{34}\ d_{56}$

$M_1\ \ M_6 =$ áll.
"áll"

G

Feladat: a) EOM felírása ált. koord.; fogaskerekek nyelfotolása

b) olyan ált. koordináta, amivel a modell láncszerű lesz

DOFOK SZÁMA: 4

ált. koord: $q_1 = \varphi_1$ $\quad \varphi_3$ és φ_5 nem független, φ_2-vel és φ_4-el adódik:

$q_2 = \varphi_2$

$q_3 = \varphi_4$

$q_4 = \varphi_6$

$$D_2 \varphi_2 = D_3 \varphi_3 \quad \Downarrow \quad \varphi_3 = \frac{D_2}{D_3}\varphi_2$$

$$D_4 \varphi_4 = D_5 \varphi_5 \quad \Downarrow \quad \varphi_5 = \frac{D_4}{D_5}\varphi_4$$

Lagrange egyenlet:

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot q_i}\right) - \frac{\partial E}{\partial q_i} = Q_{c,i} + Q_{g,i}$$

$i = 1,2,3,4$

$$\underline{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_4 \\ \varphi_6 \end{bmatrix}$$

• A rendszer teljes kinetikai energiája

$$① \quad E = \sum_{i=1}^{6} \frac{1}{2} J_i \omega_i^2 = \frac{1}{2} J_1 \omega_1^2 + \frac{1}{2} J_2 \omega_2^2 + \frac{1}{2} J_3 \omega_3^2 + \frac{1}{2} J_4 \omega_4^2 + \frac{1}{2} J_5 \omega_5^2 + \frac{1}{2} J_6 \omega_6^2 =$$

$$= \frac{1}{2} J_1 \dot{\varphi}_1^2 + \frac{1}{2} J_2 \dot{\varphi}_2^2 + \frac{1}{2} J_3 \dot{\varphi}_3^2 + \frac{1}{2} J_4 \dot{\varphi}_4^2 + \frac{1}{2} J_5 \dot{\varphi}_5^2 + \frac{1}{2} J_6 \dot{\varphi}_6^2 =$$

$$= \frac{1}{2} J_1 \dot{\varphi}_1^2 + \frac{1}{2} J_2 \dot{\varphi}_2^2 + \frac{1}{2} J_3 \left(\frac{D_2}{D_3} \dot{\varphi}_2\right)^2 + \frac{1}{2} J_4 \dot{\varphi}_4^2 + \frac{1}{2} J_5 \left(\frac{D_4}{D_5} \dot{\varphi}_4\right)^2 + \frac{1}{2} J_6 \dot{\varphi}_6^2 =$$

$$= \frac{1}{2} J_1 \dot{\varphi}_1^2 + \frac{1}{2}\left[J_2 + J_3 \frac{D_2^2}{D_3^2}\right] \dot{\varphi}_2^2 + \frac{1}{2}\left[J_4 + J_5 \frac{D_4^2}{D_5^2}\right] \dot{\varphi}_4^2 + \frac{1}{2} J_6 \dot{\varphi}_6^2$$

---

torziós rugóállandó: $\gamma = \frac{l}{I_p G}$ ahol $I_p = \frac{d^4 \pi}{32}$ $\Rightarrow \gamma = \frac{32 l}{d^4 \pi G}$

I_p: poláris 2. rendű nyomaték

$\vartheta \Rightarrow$ szögelfordulás

• A rendszer teljes rugóenergiája

$$② \quad U = U_1 + U_2 + U_3 = \frac{1}{2} \frac{\vartheta_{12}^2}{\gamma_{12}} + \frac{1}{2} \frac{\vartheta_{34}^2}{\gamma_{34}} + \frac{1}{2} \frac{\vartheta_{56}^2}{\gamma_{56}} =$$

$$= \frac{1}{2} \frac{(\varphi_2 - \varphi_1)^2}{\gamma_{12}} + \frac{1}{2} \frac{(\varphi_4 - \varphi_3)^2}{\gamma_{34}} + \frac{1}{2} \frac{(\varphi_6 - \varphi_5)^2}{\gamma_{56}} =$$

$$= \frac{1}{2} \frac{(\varphi_2 - \varphi_1)^2}{\gamma_{12}} + \frac{1}{2} \frac{\left(\varphi_4 - \frac{D_2}{D_3} \varphi_2\right)^2}{\gamma_{34}} + \frac{1}{2} \frac{\left(\varphi_6 - \frac{D_4}{D_5} \varphi_4\right)^2}{\gamma_{56}}$$

ahol tehát

$$I_{p_{i,i+1}} = \frac{d_{i,i+1}^4 \pi}{32}$$

$$\gamma_{i,i+1} = \frac{l_{i,i+1}}{I_{p_{i,i+1}} G} = \frac{32\, l_{i,i+1}}{d_{i,i+1}^4 \pi G} \qquad i = 1, 3, 5$$

③ A külső ER. teljesítménye:

$$P = M_1 \dot{\varphi}_1 + M_6 \dot{\varphi}_6$$

• Az 1. Mozgásegyenlet $\boxed{i=1} \Rightarrow q_1 = \varphi_1$ eset

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_1}\right) - \frac{\partial E}{\partial q_1} = Q_{c1} + Q_{g1}$$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\varphi}_1}\right)}_{①} - \underbrace{\frac{\partial E}{\partial \varphi_1}}_{=0} = \underbrace{Q_{c1}}_{②} + \underbrace{Q_{g1}}_{③}$$

① $\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_1}\right) = \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\varphi}_1}\right) = \Theta_1 \ddot{\varphi}_1$

② $Q_{c1} = -\frac{\partial U}{\partial q_1} = -\frac{\partial U}{\partial \varphi_1} = -\left(\frac{\varphi_2 - \varphi_1}{\gamma_{12}} \cdot (-1)\right) = -\frac{1}{\gamma_{12}}\varphi_1 + \frac{1}{\gamma_{12}}\varphi_2$

③ $Q_{q1} = \frac{\partial P}{\partial \dot{q}_1} = \frac{\partial P}{\partial \dot{\varphi}_1} = M_1$

• A 2. Mozgásegyenlet: $\boxed{i=2} \Rightarrow q_2 = \varphi_2$ eset

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_1}\right) - \frac{\partial E}{\partial q_2} = Q_{c2} + Q_{g2} \; ; \; \underbrace{\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\varphi}_2}\right)}_{①} - \underbrace{\frac{\partial E}{\partial \varphi_2}}_{=0} = \underbrace{Q_{c2}}_{②} + \underbrace{Q_{q2}}_{③}$$

① $\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_2}\right) = \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\varphi}_2}\right) = \left(\Theta_2 + \Theta_3 \frac{n_2^2}{n_3^2}\right)\ddot{\varphi}_2$

② $Q_{c2} = -\frac{\partial U}{\partial q_2} = -\frac{\partial U}{\partial \varphi_2} = -\left(\frac{\varphi_2 - \varphi_1}{\gamma_{12}} + \frac{\varphi_4 - \frac{n_2}{n_3}\varphi_2}{\gamma_{34}} \cdot \left(-\frac{n_2}{n_3}\right)\right) =$

$$= -\frac{1}{\gamma_{12}}\varphi_2 + \frac{1}{\gamma_{12}}\varphi_1 + \frac{\frac{n_2}{n_3}}{\gamma_{34}}\varphi_4 - \frac{\frac{n_2^2}{n_3^2}}{\gamma_{34}}\varphi_2 =$$

$$= \frac{1}{\gamma_{12}}\varphi_1 - \left(\frac{1}{\gamma_{12}} + \frac{n_2^2}{n_3^2 \gamma_{34}}\right)\varphi_2 + \frac{n_2}{n_3 \gamma_{34}}\varphi_4$$

③ $Q_{q2} = \frac{\partial P}{\partial \dot{q}_1} = \frac{\partial P}{\partial \dot{\varphi}_2} = 0$

• A 3. Mozgásegyenlet: $\boxed{i=3} \Rightarrow q_3 = l_4$ eset

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot q_3}\right) - \frac{\partial E}{\partial q_3} = Q_{c3} + Q_{g3} \qquad \underbrace{\frac{d}{dt}\left(\frac{\partial E}{\partial \dot l_4}\right)}_{①} - \underbrace{\frac{\partial E}{\partial l_4}}_{=0} = \underbrace{Q_{c3}}_{②} + \underbrace{Q_{g3}}_{③}$$

$$① \quad \frac{d}{dt}\left(\frac{\partial E}{\partial \dot q_3}\right) = \frac{d}{dt}\left(\frac{\partial E}{\partial \dot l_4}\right) = \left(J_4 + J_5 \frac{D_4^2}{D_5^2}\right)\ddot l_4$$

$$② \quad Q_{c3} = -\frac{\partial U}{\partial q_3} = -\frac{\partial U}{\partial l_4} = -\left(\frac{l_4 - \frac{D_2}{D_3} l_2}{\gamma_{34}} + \frac{l_6 - \frac{D_4}{D_5} l_4}{\gamma_{56}}\left(-\frac{D_4}{D_5}\right)\right) =$$

$$= -\frac{1}{\gamma_{34}} l_4 + \frac{\frac{D_2}{D_3}}{\gamma_{34}} l_2 + \frac{\frac{D_4}{D_5}}{\gamma_{56}} l_6 - \frac{\frac{D_4^2}{D_5^2}}{\gamma_{56}} l_4 =$$

$$= \frac{D_2}{D_3 \gamma_{34}} l_2 - \left(\frac{1}{\gamma_{34}} + \frac{D_4^2}{D_5^2 \gamma_{56}}\right) l_4 + \frac{D_4}{D_5 \gamma_{56}} l_6$$

$$③ \quad Q_{g3} = \frac{\partial P}{\partial \dot q_3} = \frac{\partial P}{\partial \dot l_4} = 0$$

• A 4. Mozgásegyenlet: $\boxed{i=4} \Rightarrow q_4 = l_6$ eset

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot q_4}\right) - \frac{\partial E}{\partial q_4} = Q_{c4} + Q_{g4} \qquad \underbrace{\frac{d}{dt}\left(\frac{\partial E}{\partial \dot l_6}\right)}_{①} - \underbrace{\frac{\partial E}{\partial l_6}}_{=0} = \underbrace{Q_{c4}}_{②} + \underbrace{Q_{g4}}_{③}$$

$$① \quad \frac{d}{dt}\left(\frac{\partial E}{\partial \dot q_4}\right) = \frac{d}{dt}\left(\frac{\partial E}{\partial \dot l_6}\right) = J_6 \ddot l_6$$

$$② \quad Q_{c4} = -\frac{\partial U}{\partial q_4} = -\frac{\partial U}{\partial l_6} = -\left(\frac{l_6 - \frac{D_4}{D_5} l_4}{\gamma_{56}}\right) = -\frac{1}{\gamma_{56}} l_6 + \frac{\frac{D_4}{D_5}}{\gamma_{56}} l_4 =$$

$$= \frac{D_4}{D_5 \gamma_{56}} l_4 - \frac{1}{\gamma_{56}} l_6$$

$$③ \quad Q_{g4} = \frac{\partial P}{\partial \dot q_4} = \frac{\partial P}{\partial \dot l_6} = M_6$$

· Visszahelyettesítve a Lagrange egyenletbe a mozgásegyenlet rendszer:

$$\left.\begin{aligned}
&\boxed{i=1} \Rightarrow J_1\ddot{\varphi}_1 + \frac{1}{\gamma_{12}}\varphi_1 - \frac{1}{\gamma_{12}}\varphi_2 = M_1 \\
&\boxed{i=2} \Rightarrow \left(J_2 + J_3\frac{D_2^2}{D_3^2}\right)\ddot{\varphi}_2 - \frac{1}{\gamma_{12}}\varphi_1 + \left(\frac{1}{\gamma_{12}} + \frac{D_2^2}{D_3^2\gamma_{34}}\right)\varphi_2 - \frac{D_2}{D_3\gamma_{34}}\varphi_4 = 0 \\
&\boxed{i=3} \Rightarrow \left(J_4 + J_5\frac{D_4^2}{D_5^2}\right)\ddot{\varphi}_4 - \frac{D_2}{D_3\gamma_{34}}\varphi_2 + \left(\frac{1}{\gamma_{34}} + \frac{D_4^2}{D_5^2\gamma_{56}}\right)\varphi_4 - \frac{D_4}{D_5\gamma_{56}}\varphi_6 = 0 \\
&\boxed{i=4} \Rightarrow J_6\ddot{\varphi}_6 - \frac{D_4}{D_5\gamma_{56}}\varphi_4 + \frac{1}{\gamma_{56}}\varphi_6 = M_6
\end{aligned}\right\}$$

Mátrixos alakban:

$$\overbrace{\begin{bmatrix} J_1 & 0 & 0 & 0 \\ 0 & J_2 + J_3\frac{D_2^2}{D_3^2} & 0 & 0 \\ 0 & 0 & J_4 + J_5\frac{D_4^2}{D_5^2} & 0 \\ 0 & 0 & 0 & J_6 \end{bmatrix}}^{\underline{\underline{M}}} \overbrace{\begin{bmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \\ \ddot{\varphi}_4 \\ \ddot{\varphi}_6 \end{bmatrix}}^{\underline{\ddot{q}}} +$$

$$+ \underbrace{\begin{array}{cccc} \varphi_1 & \varphi_2 & \varphi_4 & \varphi_6 \\ \left[\begin{matrix} \frac{1}{\gamma_{12}} & -\frac{1}{\gamma_{12}} & 0 & 0 \\ -\frac{1}{\gamma_{12}} & \frac{1}{\gamma_{12}} + \frac{D_2^2}{D_3^2\gamma_{34}} & -\frac{D_2}{D_3\gamma_{34}} & 0 \\ 0 & -\frac{D_2}{D_3\gamma_{34}} & \frac{1}{\gamma_{34}} + \frac{D_4^2}{D_5^2\gamma_{56}} & -\frac{D_4}{D_5\gamma_{56}} \\ 0 & 0 & -\frac{D_4}{D_5\gamma_{56}} & \frac{1}{\gamma_{56}} \end{matrix}\right] \end{array}}_{\underline{\underline{K}}} \underbrace{\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_4 \\ \varphi_6 \end{bmatrix}}_{\underline{q}} = \underbrace{\begin{bmatrix} M_1 \\ 0 \\ 0 \\ M_6 \end{bmatrix}}_{\underline{Q_q}}$$

• lúncszerűség feltételei:

$\underline{\underline{M}}$ diagonális ✓

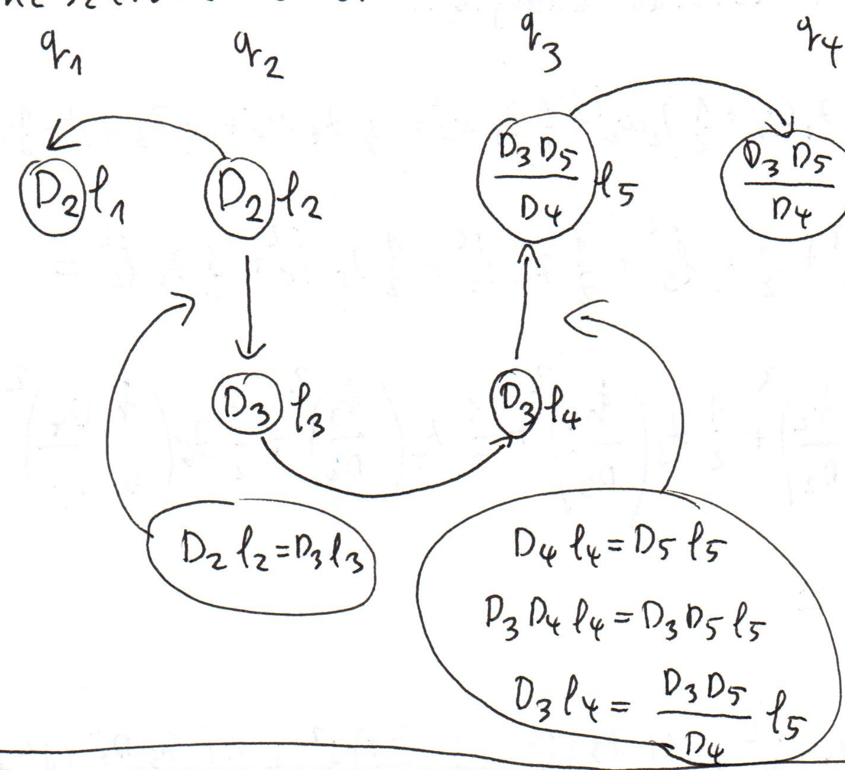
$\det(\underline{\underline{M}}) > 0$ ✓

$\underline{\underline{C}}$ • kodiagonális ✓

• szimmetrikus ✓

• sorok, oszlopok összege 0 ✗

lánc szerűvé tétel



tehát:

$$\boxed{\begin{aligned} q_1 &= D_2 l_1 \\ q_2 &= D_2 l_2 = D_3 l_3 \\ q_3 &= D_3 l_4 = \frac{D_3 D_5}{D_4} l_5 \\ q_4 &= \frac{D_3 D_5}{D_4} l_6 \end{aligned}}$$

$$l_1 = \frac{q_1}{D_2} \qquad l_5 = \frac{q_3 D_4}{D_3 D_5}$$

$$l_2 = \frac{q_2}{D_2} \qquad l_6 = \frac{q_4 D_4}{D_3 D_5}$$

$$l_3 = \frac{q_2}{D_3}$$

$$l_4 = \frac{q_3}{D_3}$$

$$\Downarrow$$

$$\underline{\underline{q}} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} D_2 l_1 \\ D_2 l_2 \\ D_3 l_4 \\ \frac{D_3 D_5}{D_4} l_6 \end{bmatrix}$$

Lagrange egyenletei:

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_i}\right) - \frac{\partial E}{\partial q_i} = Q_{ci} + Q_{qi}$$

$$i = 1, 2, 3, 4$$

① A rendszer teljes kinetikai energiája

$$E=\sum_{i=1}^{6}\frac{1}{2}J_i\omega_i^2=\frac{1}{2}J_1\omega_1^2+\frac{1}{2}J_2\omega_2^2+\frac{1}{2}J_3\omega_3^2+\frac{1}{2}J_4\omega_4^2+\frac{1}{2}J_5\omega_5^2+\frac{1}{2}J_6\omega_6^2=$$

$$=\frac{1}{2}J_1\dot{\varphi}_1^2+\frac{1}{2}J_2\dot{\varphi}_2^2+\frac{1}{2}J_3\dot{\varphi}_3^2+\frac{1}{2}J_4\dot{\varphi}_4^2+\frac{1}{2}J_5\dot{\varphi}_5^2+\frac{1}{2}J_6\dot{\varphi}_6^2=$$

$$=\frac{1}{2}J_1\left(\frac{\dot{q}_1}{D_2}\right)^2+\frac{1}{2}J_2\left(\frac{\dot{q}_2}{D_2}\right)^2+\frac{1}{2}J_3\left(\frac{\dot{q}_2}{D_3}\right)^2+\frac{1}{2}J_4\left(\frac{\dot{q}_3}{D_3}\right)^2+\frac{1}{2}J_5\left(\frac{\dot{q}_3D_4}{D_3D_5}\right)^2+$$

$$+\frac{1}{2}J_6\left(\frac{\dot{q}_4D_4}{D_3D_5}\right)^2=$$

$$=\frac{1}{2}\left(\frac{J_1}{D_2^2}\right)\dot{q}_1^2+\frac{1}{2}\left(\frac{J_2}{D_2^2}\right)\dot{q}_2^2+\frac{1}{2}\left(\frac{J_3}{D_3^2}\right)\dot{q}_2^2+\frac{1}{2}\left(\frac{J_4}{D_3^2}\right)\dot{q}_3^2+\frac{1}{2}\left(\frac{J_5D_4^2}{D_3^2D_5^2}\right)\dot{q}_3^2+$$

$$+\frac{1}{2}\left(\frac{J_6D_4^2}{D_3^2D_5^2}\right)\dot{q}_4^2=$$

$$=\frac{1}{2}\left(\frac{J_1}{D_2^2}\right)\dot{q}_1^2+\frac{1}{2}\left(\frac{J_2}{D_2^2}+\frac{J_3}{D_3^2}\right)\dot{q}_2^2+\frac{1}{2}\left(\frac{J_4}{D_3^2}+\frac{J_5D_4^2}{D_3^2D_5^2}\right)\dot{q}_3^2+\frac{1}{2}\left(\frac{J_6D_4^2}{D_3^2D_5^2}\right)\dot{q}_4^2$$

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② A rendszer teljes rugóenergiája:

$$U=U_1+U_2+U_3=\frac{1}{2}\frac{\vartheta_{12}^2}{\gamma_{12}}+\frac{1}{2}\frac{\vartheta_{34}^2}{\gamma_{34}}+\frac{1}{2}\frac{\vartheta_{56}^2}{\gamma_{56}}=$$

$$=\frac{1}{2}\frac{(\varphi_2-\varphi_1)^2}{\gamma_{12}}+\frac{1}{2}\frac{(\varphi_4-\varphi_3)^2}{\gamma_{34}}+\frac{1}{2}\frac{(\varphi_6-\varphi_5)^2}{\gamma_{56}}=$$

$$=\frac{1}{2}\frac{\left(\frac{q_2}{D_2}-\frac{q_1}{D_2}\right)^2}{\gamma_{12}}+\frac{1}{2}\frac{\left(\frac{q_3}{D_3}-\frac{q_2}{D_3}\right)^2}{\gamma_{34}}+\frac{1}{2}\frac{\left(\frac{q_4D_4}{D_3D_5}-\frac{q_3D_4}{D_3D_5}\right)^2}{\gamma_{56}}=$$

$$=\frac{1}{2}\frac{(q_2-q_1)^2}{D_2^2\gamma_{12}}+\frac{1}{2}\frac{(q_3-q_2)^2}{D_3^2\gamma_{34}}+\frac{1}{2}\frac{D_4^2(q_4-q_3)^2}{D_3^2D_5^2\gamma_{56}}$$

③ A külső ER. teljesítménye:

$$P = M_1 \omega_1 + M_6 \omega_6 = M_1 \dot{\varphi}_1 + M_6 \dot{\varphi}_6 = M_1 \frac{\dot{q}_1}{n_2} + M_6 \frac{\dot{q}_4 n_4}{n_3 n_5} =$$

$$= \frac{M_1}{n_2} \dot{q}_1 + \frac{M_6 n_4}{n_3 n_5} \dot{q}_4$$

Az 1. Mozgásegyenlet $\boxed{i=1}$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_1}\right)}_{①} - \underbrace{\frac{\partial E}{\partial q_1}}_{=0} = \underbrace{Q_{c_1}}_{②} + \underbrace{Q_{q_1}}_{③}$$

① $\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_1}\right) = \frac{J_1}{n_2^2} \ddot{q}_1$

② $Q_{c_1} = -\frac{\partial U}{\partial q_1} = -\left(\frac{q_2 - q_1}{n_2^2 \gamma_{12}} \cdot (-1)\right) = -\frac{1}{n_2^2 \gamma_{12}} q_1 + \frac{1}{n_2^2 \gamma_{12}} q_2$

③ $Q_{q_1} = \frac{\partial P}{\partial \dot{q}_1} = \frac{M_1}{n_2}$

A 2. Mozgásegyenlet $\boxed{i=2}$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_2}\right)}_{①} - \underbrace{\frac{\partial E}{\partial q_2}}_{=0} = \underbrace{Q_{c_2}}_{②} + \underbrace{Q_{q_2}}_{③}$$

① $\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_2}\right) = \left(\frac{J_2}{n_2^2} + \frac{J_3}{n_3^2}\right) \ddot{q}_2$

② $Q_{c_2} = -\frac{\partial U}{\partial q_2} = -\left(\frac{q_2 - q_1}{n_2^2 \gamma_{12}} + \frac{q_3 - q_2}{n_3^2 \gamma_{34}} \cdot (-1)\right) = \frac{1}{n_2^2 \gamma_{12}} q_1 - \frac{1}{n_2^2 \gamma_{12}} q_2 +$

$+ \frac{1}{n_3^2 \gamma_{34}} q_3 - \frac{1}{n_3^2 \gamma_{34}} q_2 = \frac{1}{n_2^2 \gamma_{12}} q_1 - \left(\frac{1}{n_2^2 \gamma_{12}} + \frac{1}{n_3^2 \gamma_{34}}\right) q_2 + \frac{1}{n_3^2 \gamma_{34}} q_3$

③ $Q_{q_2} = \frac{\partial P}{\partial \dot{q}_2} = 0$

• A 3. Mozgásegyenlet $\boxed{i=3}$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_3}\right)}_{①} - \underbrace{\frac{\partial E}{\partial q_3}}_{=0} = \underbrace{Q_{c3}}_{②} + \underbrace{Q_{q3}}_{③}$$

① $$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_3}\right) = \left(\frac{J_4}{n_4^2} + \frac{J_5 n_4^2}{n_3^2 n_5^2}\right)\ddot{q}_3$$

② $$Q_{c3} = -\frac{\partial U}{\partial q_3} = -\left(\frac{q_3 - q_2}{n_3^2 \gamma_{34}} + \frac{n_4^2 (q_4 - q_3)}{n_3^2 n_5^2 \gamma_{56}} \cdot (-1)\right) =$$

$$= -\frac{1}{n_3^2 \gamma_{34}} q_3 + \frac{1}{n_3^2 \gamma_{34}} q_2 + \frac{n_4^2}{n_3^2 n_5^2 \gamma_{56}} q_4 - \frac{n_4^2}{n_3^2 n_5^2 \gamma_{56}} q_3 =$$

$$= \frac{1}{n_3^2 \gamma_{34}} q_2 - \left(\frac{1}{n_3^2 \gamma_{34}} + \frac{n_4^2}{n_3^2 n_5^2 \gamma_{56}}\right) q_3 + \frac{n_4^2}{n_3^2 n_5^2 \gamma_{56}} q_4$$

③ $$Q_{q3} = \frac{\partial P}{\partial \dot{q}_3} = 0$$

A 4. Mozgásegyenlet: $\boxed{i=4}$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_4}\right)}_{①} - \underbrace{\frac{\partial E}{\partial q_4}}_{=0} = \underbrace{Q_{c4}}_{②} + \underbrace{Q_{q4}}_{③}$$

① $$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_4}\right) = \frac{J_6 n_4^2}{n_3^2 n_5^2} \ddot{q}_4$$

② $$Q_{c4} = -\frac{\partial U}{\partial q_4} = -\left(\frac{n_4^2 (q_4 - q_3)}{n_3^2 n_5^2 \gamma_{56}}\right) = \frac{n_4^2}{n_3^2 n_5^2 \gamma_{56}} q_3 - \frac{n_4^2}{n_3^2 n_5^2 \gamma_{56}} q_4$$

③ $$Q_{q4} = \frac{\partial P}{\partial \dot{q}_4} = \frac{M_6 n_4}{n_3 n_5}$$

Visszahelyettesítve a mozgásegyenletrendszer:

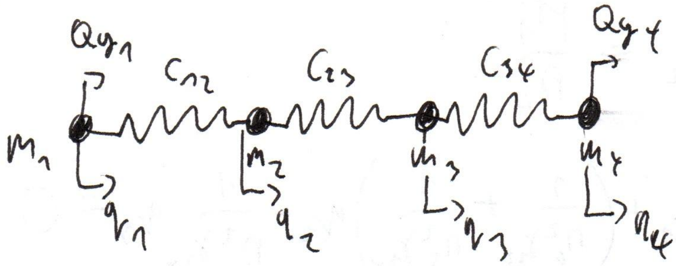
$$\left.\begin{aligned}
&\boxed{i=1} \rightarrow \frac{J_1}{n_2^2}\ddot{q}_1 + \frac{1}{n_2^2\gamma_{12}}q_1 - \frac{1}{n_2^2\gamma_{12}}q_2 = \frac{M_1}{n_2}\\
&\boxed{i=2} \rightarrow \left(\frac{J_2}{n_2^2}+\frac{J_3}{n_3^2}\right)\ddot{q}_2 - \frac{1}{n_2^2\gamma_{12}}q_1 + \left(\frac{1}{n_2^2\gamma_{12}}+\frac{1}{n_3^2\gamma_{34}}\right)q_2 - \frac{1}{n_3^2\gamma_{34}}q_3 = 0\\
&\boxed{i=3} \rightarrow \left(\frac{J_4}{n_4^2}+\frac{J_5 n_4^2}{n_3^2 n_5^2}\right)\ddot{q}_3 - \frac{1}{n_3^2\gamma_{34}}q_2 + \left(\frac{1}{n_3^2\gamma_{34}}+\frac{n_4^2}{n_3^2n_5^2\gamma_{56}}\right)q_3 - \frac{n_4^2}{n_3^2n_5^2\gamma_{56}}q_4 = 0\\
&\boxed{i=4} \rightarrow \frac{J_6 n_4^2}{n_3^2n_5^2}\ddot{q}_4 - \frac{n_4^2}{n_3^2n_5^2\gamma_{56}}q_3 + \frac{n_4^2}{n_3^2n_5^2\gamma_{56}}q_4 = \frac{M_6 n_4}{n_3 n_5}
\end{aligned}\right\}$$

Mátrixos alakban:

$$\underbrace{\begin{bmatrix}
\frac{J_1}{n_2^2} & 0 & 0 & 0\\
0 & \frac{J_2}{n_2^2}+\frac{J_3}{n_3^2} & 0 & 0\\
0 & 0 & \frac{J_4}{n_3^2}+\frac{n_4^2 J_5}{n_3^2 n_5^2} & 0\\
0 & 0 & 0 & \frac{n_4^2 J_6}{n_3^2 n_5}
\end{bmatrix}}_{\underline{\underline{M}}}
\underbrace{\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\\ \ddot{q}_3\\ \ddot{q}_4\end{bmatrix}}_{\underline{\ddot{q}}} +$$

$$+\underbrace{\begin{bmatrix}
\frac{1}{n_2^2\gamma_{12}} & -\frac{1}{n_2^2\gamma_{12}} & 0 & 0\\
-\frac{1}{n_2^2\gamma_{12}} & \frac{1}{n_2^2\gamma_{12}}+\frac{1}{n_3^2\gamma_{34}} & -\frac{1}{n_3^2\gamma_{34}} & 0\\
0 & -\frac{1}{n_3^2\gamma_{34}} & \frac{1}{n_3^2\gamma_{34}}+\frac{n_4^2}{n_3^2n_5^2\gamma_{56}} & -\frac{n_4^2}{n_3^2n_5^2\gamma_{56}}\\
0 & 0 & -\frac{n_4^2}{n_3^2n_5^2\gamma_{56}} & \frac{n_4^2}{n_3^2n_5^2\gamma_{56}}
\end{bmatrix}}_{\underline{\underline{K}}}
\underbrace{\begin{bmatrix}q_1\\ q_2\\ q_3\\ q_4\end{bmatrix}}_{\underline{q}} =
\underbrace{\begin{bmatrix}\frac{M_1}{n_2}\\ 0\\ 0\\ \frac{M_6 n_4}{n_3 n_5}\end{bmatrix}}_{\underline{Q}_q}$$

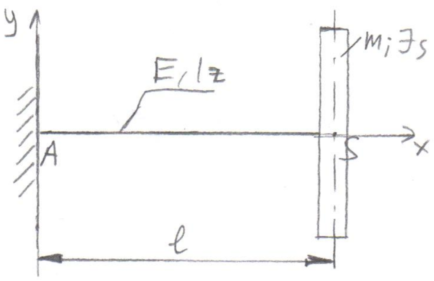
A láncszerű modell:



$$m_1 = \frac{\Theta_1}{n_3^2} \qquad m_2 = \frac{\Theta_2}{n_2^2} + \frac{\Theta_3}{n_3^2} \qquad m_3 = \frac{\Theta_4}{n_3^2} + \frac{n_4^2 \Theta_5}{n_3^2 n_5^2} \qquad m_4 = \frac{n_4^2 \Theta_6}{n_3^2 n_5^2}$$

$$\frac{1}{c_{12}} = \frac{1}{n_2^2 \gamma_{12}} \qquad \frac{1}{c_{23}} = \frac{1}{n_3^2 \gamma_{34}} \qquad \frac{1}{c_{34}} = \frac{n_4^2}{n_3^2 n_5^2 \gamma_{56}}$$

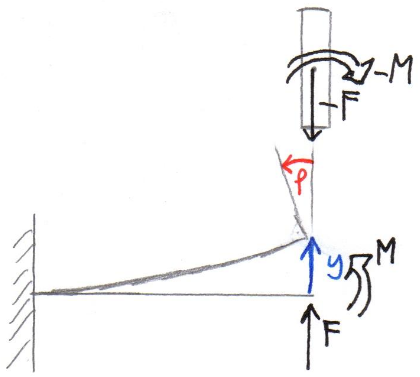
$$Q_{q_1} = \frac{M_1}{n_2} \qquad Q_{q_4} = \frac{M_6 n_4}{n_3 n_5}$$



adott: m, J_S, E, I_z, ℓ

feladat: • rezgő rendszer mozgásegyenlete
• Maxwell-féle hatásszámok

• általános koordináták: • y: fogaskerék függőleges elmozdulása
• φ: fogaskerék z tengely körüli szögelfordulása



F: a fogaskerékről a tengelyre ható erő
M: a fogaskerékről a tengelyre ható nyomaték
$-F$: a tengelyről a fogaskerékre ható erő
$-M$: a tengelyről a fogaskerékre ható nyomaték

1) impulzus tétel: $-F = m\ddot{y}$

2) perdület tétel: $-M = J_S\ddot{\varphi}$

3) a függőleges elmozdulás: $y = \delta F + \mu M$

4) a z tengely körüli szögelfordulás: $\varphi = \chi F + \kappa M$

ahol:
δ: egységnyi erő hatására létrejövő y elmozdulás
μ: egységnyi nyomaték hatására létrejövő y elmozdulás
χ: egységnyi erő hatására létrejövő φ szögelfordulás
κ: egységnyi nyomaték hatására létrejövő φ szögelfordulás

Maxwell-féle hatásszámok

• 1)-t és 2)-t behelyettesítve 3)-ba és 4)-be:

$$\left.\begin{aligned} y + \delta m\ddot{y} + \mu J_S\ddot{\varphi} &= 0 \\ \varphi + \chi m\ddot{y} + \kappa J_S\ddot{\varphi} &= 0 \end{aligned}\right\}$$

• mátrixos alakban felírva:

$$\underbrace{\begin{array}{cc} \ddot{y} & \ddot{\varphi} \\ \end{array}}_{}\qquad \underbrace{\begin{array}{cc} y & \varphi \\ \end{array}}_{}$$

$$\underbrace{\begin{bmatrix} \delta m & \mu J_S \\ \chi m & \kappa J_S \end{bmatrix}}_{\underline{\underline{M}}} \underbrace{\begin{bmatrix} \ddot{y} \\ \ddot{\varphi} \end{bmatrix}}_{\ddot{\underline{q}}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\underline{\underline{E}}} \underbrace{\begin{bmatrix} y \\ \varphi \end{bmatrix}}_{\underline{q}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\underline{0}}$$

$\underline{\underline{m}}$ = módosított tömegmátrix, ez 2 részre bontható:

$$\underline{\underline{m}} = \underbrace{\begin{bmatrix} \delta & \mu \\ \chi & \varkappa \end{bmatrix}}_{\underline{\underline{D}}} \underbrace{\begin{bmatrix} m & 0 \\ 0 & J_s \end{bmatrix}}_{\underline{\underline{M}}}$$

$\underline{\underline{M}}$ ↳ tömegmátrix

$\underline{\underline{D}}$ ↳ Maxwell-féle hatásszámok mátrixa

Így a mozgásegyenlet:

$$\underbrace{\begin{bmatrix} \delta & \mu \\ \chi & \varkappa \end{bmatrix}}_{\underline{\underline{D}}} \underbrace{\begin{bmatrix} m & 0 \\ 0 & J_s \end{bmatrix}}_{\underline{\underline{M}}} \underbrace{\begin{bmatrix} \ddot{y} \\ \ddot{\varphi} \end{bmatrix}}_{\ddot{\underline{q}}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\underline{\underline{E}}} \underbrace{\begin{bmatrix} y \\ \varphi \end{bmatrix}}_{\underline{q}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\underline{0}}$$

$$\underline{\underline{D}}\,\underline{\underline{M}}\,\ddot{\underline{q}} + \underline{\underline{E}}\,\underline{q} = \underline{0} \qquad / \cdot \underline{\underline{D}}^{-1}$$

megjegyzés:

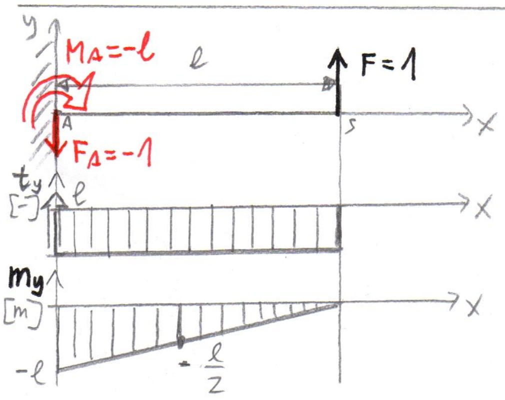
$$\underbrace{\underline{\underline{D}}^{-1}\underline{\underline{D}}}_{\underline{\underline{E}}}\,\underline{\underline{M}}\,\ddot{\underline{q}} + \underbrace{\underline{\underline{D}}^{-1}}_{\underline{\underline{K}}}\underline{\underline{E}}\,\underline{q} = \underline{0}$$

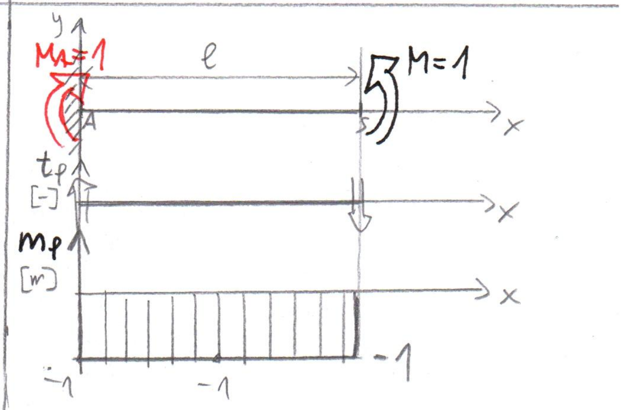
$$\underline{\underline{M}}\,\ddot{\underline{q}} + \underline{\underline{K}}\,\underline{q} = \underline{0}$$

ahol: $\underline{\underline{K}} = \underline{\underline{D}}^{-1} = \dfrac{\operatorname{adj} \underline{\underline{D}}}{\det \underline{\underline{D}}} = \dfrac{\begin{bmatrix} \varkappa & -\chi \\ -\mu & \delta \end{bmatrix}}{\delta \cdot \varkappa - \chi \mu}$

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & J_s \end{bmatrix}}_{\underline{\underline{M}}} \underbrace{\begin{bmatrix} \ddot{y} \\ \ddot{\varphi} \end{bmatrix}}_{\ddot{\underline{q}}} + \underbrace{\frac{1}{\delta \cdot \varkappa - \chi \mu} \begin{bmatrix} \varkappa & -\chi \\ -\mu & \delta \end{bmatrix}}_{\underline{\underline{K}}} \underbrace{\begin{bmatrix} y \\ \varphi \end{bmatrix}}_{\underline{q}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\underline{0}}$$

• Maxwell-féle hatásszámok meghatározása Castigliano-tétel alkalmazásával





• A hajlító igénybevétel: $M_{hz} = F\cdot m_y + M m_\varphi$

• Az alakváltozási energia:

$$U = \int_{(\ell)} \frac{M_{hz}^2}{2 I_z E}\, dx = \int_{(\ell)} \frac{(F\cdot m_y + M\cdot m_\varphi)^2}{2 I_z E}\, dx$$

• A Castigliano tétel alapján az y függőleges elmozdulás:

$$y = \frac{\partial U}{\partial F} = \int_{(\ell)} \frac{F\cdot m_y + M m_\varphi}{I_z\cdot E}\cdot m_y\, dx = F\cdot \underbrace{\int_{(\ell)} \frac{m_y^2}{I_z E}\, dx}_{\delta} + M \underbrace{\int_{(\ell)} \frac{m_y m_\varphi}{I_z E}\, dx}_{\mu}$$

• A Castigliano tétel alapján a φ, z tengely körüli szög elfordulás:

$$\varphi = \frac{\partial U}{\partial M} = \int_{(\ell)} \frac{F\cdot m_y + M m_\varphi}{I_z E}\cdot m_\varphi\, dx = F\cdot \underbrace{\int_{(\ell)} \frac{m_y m_\varphi}{I_z E}\, dx}_{\varkappa} + M \underbrace{\int_{(\ell)} \frac{m_\varphi^2}{I_z E}\, dx}_{\varkappa}$$

• A Maxwell-féle hatásszámok kiszámítása Simpsonnal:

$$\delta = \int_{(\ell)} \frac{m_y^2}{I_z E}\,dx = \frac{1}{I_z E}\int_{(\ell)} m_y^2\,dx = \frac{1}{I_z E}\cdot\frac{\ell}{6}\cdot\left((-\ell)^2 + 4\cdot\left(-\frac{\ell}{2}\right)^2 + 0^2\right) =$$

$$= \frac{1}{I_z E}\cdot\frac{\ell}{6}\cdot 2\ell^2 = \frac{\ell^3}{3 I_z E}$$

$$\mu = \chi = \int_{(\ell)} \frac{m_y m_\varphi}{I_z E}\,dx = \frac{1}{I_z E}\int_{(\ell)} m_y m_\varphi\,dx = \frac{1}{I_z E}\cdot\frac{\ell}{6}\cdot\left((-\ell)(-1) + 4\cdot\left(-\frac{\ell}{2}\right)\cdot(-1) + 0(-1)\right) =$$

$$= \frac{1}{I_z E}\cdot\frac{\ell}{6}\cdot 3\ell = \frac{\ell^2}{2 I_z E}$$

$$\kappa = \int_{(\ell)} \frac{m_\varphi^2}{I_z E}\,dx = \frac{1}{I_z E}\int_{(\ell)} m_\varphi^2\,dx = \frac{1}{I_z E}\cdot\frac{\ell}{6}\cdot\left((-1)^2 + 4\cdot(-1)^2 + (-1)^2\right) =$$

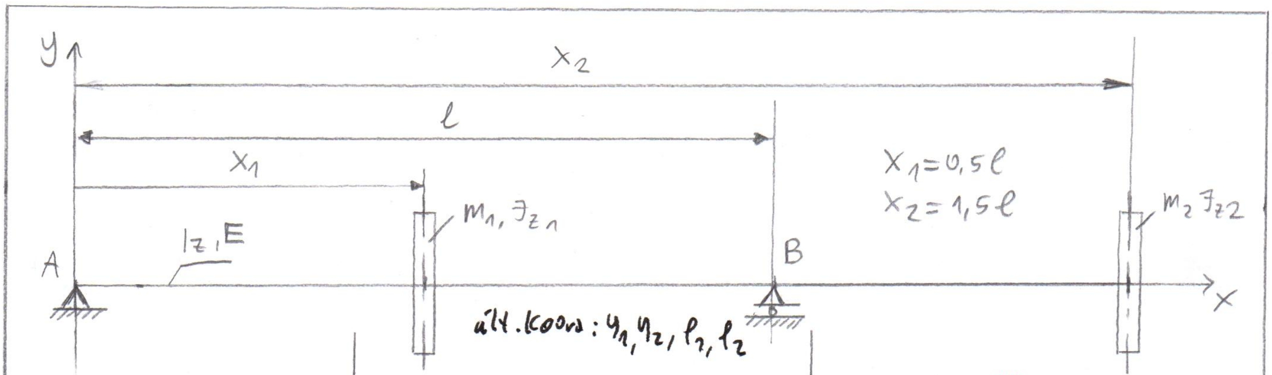
$$= \frac{1}{I_z E}\cdot\frac{\ell}{6}\cdot 6 = \frac{\ell}{I_z E}$$

• Így a Maxwell-féle hatásszámok mátrixa:

$$\underline{\underline{D}} = \begin{bmatrix} \delta & \mu \\ \chi & \kappa \end{bmatrix} = \begin{bmatrix} \frac{\ell^3}{3I_zE} & \frac{\ell^2}{2I_zE} \\ \frac{\ell^2}{2I_zE} & \frac{\ell}{I_zE} \end{bmatrix}$$

• Így a mozgásegyenlet:

$$\underbrace{\begin{bmatrix} \frac{\ell^3}{3I_zE} & \frac{\ell^2}{2I_zE} \\ \frac{\ell^2}{2I_zE} & \frac{\ell}{I_zE} \end{bmatrix}}_{\underline{\underline{D}}} \underbrace{\begin{bmatrix} m & 0 \\ 0 & J_s \end{bmatrix}}_{\underline{\underline{M}}} \underbrace{\begin{bmatrix} \ddot{y} \\ \ddot{\varphi} \end{bmatrix}}_{\underline{\ddot{q}}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\underline{\underline{E}}} \underbrace{\begin{bmatrix} y \\ \varphi \end{bmatrix}}_{\underline{q}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\underline{0}}$$



1) impulzus tétel:	$-F_1 = m_1 \ddot{y}_1$	$-F_2 = m_2 \ddot{y}_2$
2) perdület tétel:	$-M_1 = J_{z1} \ddot{\varphi}_1$	$-M_2 = J_{z2} \ddot{\varphi}_2$
3) Függőleges elmozdulás:	$y_1 = \delta_{11} F_1 + \delta_{12} F_2 + \mu_{11} M_1 + \mu_{12} M_2$	$y_2 = \delta_{21} F_1 + \delta_{22} F_2 + \mu_{21} M_1 + \mu_{22} M_2$
4) z körüli szögelfordulás:	$\varphi_1 = \chi_{11} F_1 + \chi_{12} F_2 + \kappa_{11} M_1 + \kappa_{12} M_2$	$\varphi_2 = \chi_{21} F_1 + \chi_{22} F_2 + \kappa_{21} M_1 + \kappa_{22} M_2$

ahol:

$$\underbrace{\begin{cases} \delta_{ij} = y_i \text{ elmozdulás egységnyi } F_j \text{ erő hatására} \\ \mu_{ij} = y_i \text{ elmozdulás egységnyi } M_j \text{ nyomaték hatására} \\ \chi_{ij} = \varphi_i \text{ szögelfordulás egységnyi } F_j \text{ erő hatására} \\ \kappa_{ij} = \varphi_i \text{ szögelfordulás egységnyi } M_j \text{ nyomaték hatására} \end{cases}}_{}$$

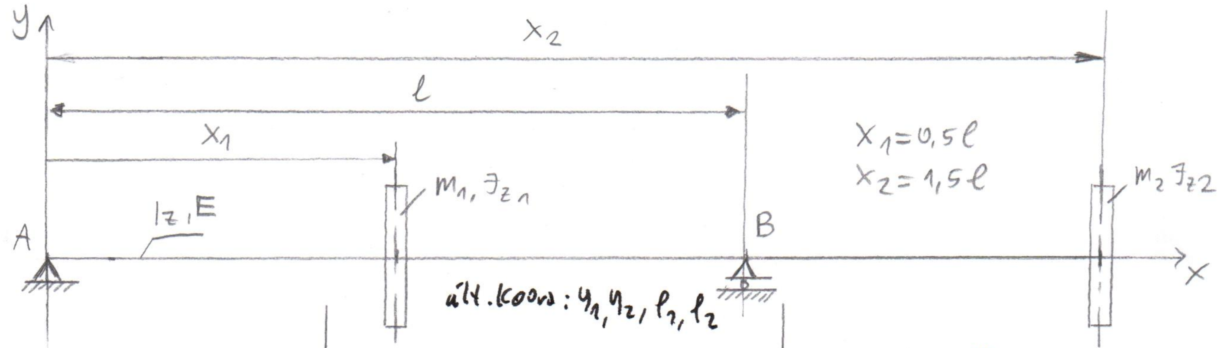
Maxwell-féle hatásszámok

- 1)-t és 2)-t behelyettesítve 3)-ba és 4)-be:

$$\left.\begin{aligned}
y_1 + \delta_{11} m_1 \ddot{y}_1 + \delta_{12} m_2 \ddot{y}_2 + \mu_{11} J_{z1} \ddot{\varphi}_1 + \mu_{12} J_{z2} \ddot{\varphi}_2 &= 0 \\
y_2 + \delta_{21} m_1 \ddot{y}_1 + \delta_{22} m_2 \ddot{y}_2 + \mu_{21} J_{z1} \ddot{\varphi}_1 + \mu_{22} J_{z2} \ddot{\varphi}_2 &= 0 \\
\varphi_1 + \chi_{11} m_1 \ddot{y}_1 + \chi_{12} m_2 \ddot{y}_2 + \kappa_{11} J_{z1} \ddot{\varphi}_1 + \kappa_{12} J_{z2} \ddot{\varphi}_2 &= 0 \\
\varphi_2 + \chi_{21} m_1 \ddot{y}_1 + \chi_{22} m_2 \ddot{y}_2 + \kappa_{21} J_{z1} \ddot{\varphi}_1 + \kappa_{22} J_{z2} \ddot{\varphi}_2 &= 0
\end{aligned}\right\}$$

- mátrixos alakba átírva:

$$\underbrace{\begin{bmatrix} \delta_{11} m_1 & \delta_{12} m_2 & \mu_{11} J_{z1} & \mu_{12} J_{z2} \\ \delta_{21} m_1 & \delta_{22} m_2 & \mu_{21} J_{z1} & \mu_{22} J_{z2} \\ \chi_{11} m_1 & \chi_{12} m_2 & \kappa_{11} J_{z1} & \kappa_{12} J_{z2} \\ \chi_{21} m_1 & \chi_{22} m_2 & \kappa_{21} J_{z1} & \kappa_{22} J_{z2} \end{bmatrix}}_{\underline{\underline{M}}} \underbrace{\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{bmatrix}}_{\underline{\ddot{q}}} + \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\underline{\underline{E}}} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \varphi_1 \\ \varphi_2 \end{bmatrix}}_{\underline{q}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{0}}$$



ált. koord: $y_1, y_2, \varphi_1, \varphi_2$

1) impulzus tétel:	$-F_1 = m_1 \ddot{y}_1$	$-F_2 = m_2 \ddot{y}_2$
2) perdület tétel:	$-M_1 = \Theta_{z1} \ddot{\varphi}_1$	$-M_2 = \Theta_{z2} \ddot{\varphi}_2$
3) Függőleges elmozdulás:	$y_1 = \delta_{11} F_1 + \delta_{12} F_2 + \mu_{11} M_1 + \mu_{12} M_2$	$y_2 = \delta_{21} F_1 + \delta_{22} F_2 + \mu_{21} M_1 + \mu_{22} M_2$
4) z körüli szögelfordulás:	$\varphi_1 = \chi_{11} F_1 + \chi_{12} F_2 + \kappa_{11} M_1 + \kappa_{12} M_2$	$\varphi_2 = \chi_{21} F_1 + \chi_{22} F_2 + \chi_{21} M_1 + \kappa_{22} M_2$

ahol:
$$\left\{\begin{array}{l} \delta_{ij} = y_i \text{ elmozdulás egységnyi } F_j \text{ erő hatására} \\ \mu_{ij} = y_i \text{ elmozdulás egységnyi } M_j \text{ nyomaték hatására} \\ \chi_{ij} = \varphi_i \text{ szögelfordulás egységnyi } F_j \text{ erő hatására} \\ \kappa_{ij} = \varphi_i \text{ szögelfordulás egységnyi } M_j \text{ nyomaték hatására} \end{array}\right.$$

→ Maxwell-féle hatásszámok

• 1)-t és 2)-t behelyettesítve 3)-ba és 4)-be:

$$\left.\begin{array}{l} y_1 + \delta_{11} m_1 \ddot{y}_1 + \delta_{12} m_2 \ddot{y}_2 + \mu_{11} \Theta_{z1} \ddot{\varphi}_1 + \mu_{12} \Theta_{z2} \ddot{\varphi}_2 = 0 \\ y_2 + \delta_{21} m_1 \ddot{y}_1 + \delta_{22} m_2 \ddot{y}_2 + \mu_{21} \Theta_{z1} \ddot{\varphi}_1 + \mu_{22} \Theta_{z2} \ddot{\varphi}_2 = 0 \\ \varphi_1 + \chi_{11} m_1 \ddot{y}_1 + \chi_{12} m_2 \ddot{y}_2 + \kappa_{11} \Theta_{z1} \ddot{\varphi}_1 + \kappa_{12} \Theta_{z2} \ddot{\varphi}_2 = 0 \\ \varphi_2 + \chi_{21} m_1 \ddot{y}_1 + \chi_{22} m_2 \ddot{y}_2 + \kappa_{21} \Theta_{z1} \ddot{\varphi}_1 + \kappa_{22} \Theta_{z2} \ddot{\varphi}_2 = 0 \end{array}\right\}$$

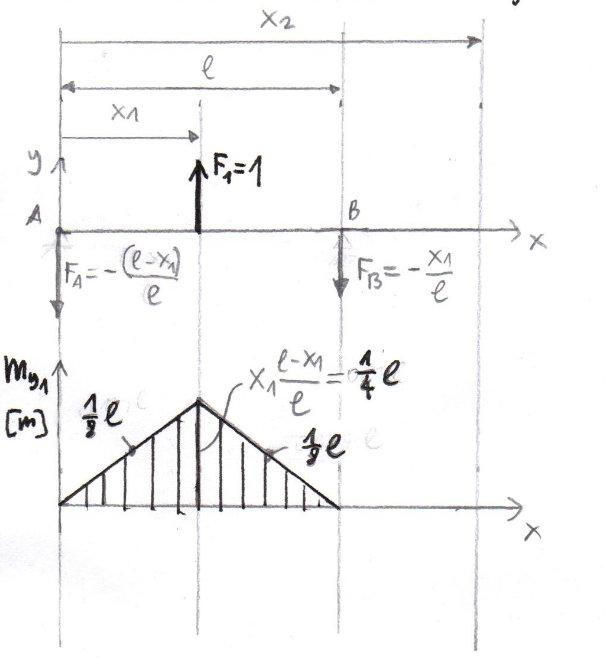
• mátrixos alakba átírva:

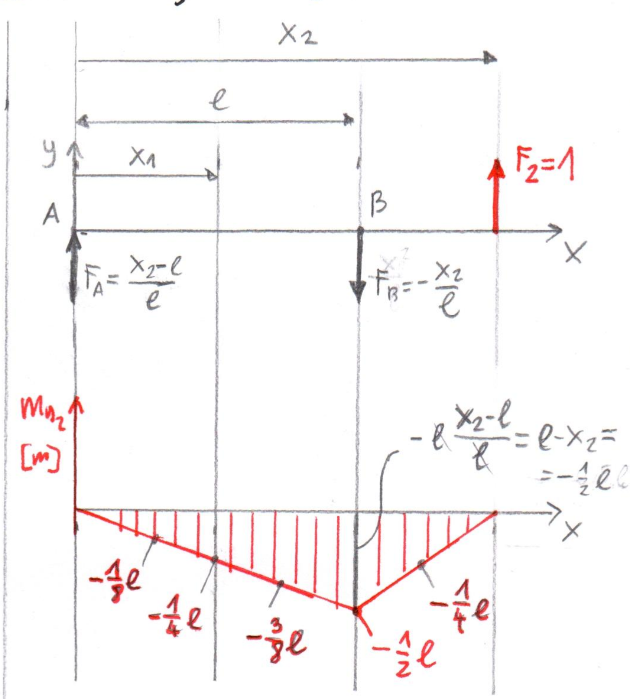
$$\underbrace{\begin{bmatrix} \delta_{11} m_1 & \delta_{12} m_2 & \mu_{11} \Theta_{z1} & \mu_{12} \Theta_{z2} \\ \delta_{21} m_1 & \delta_{22} m_2 & \mu_{21} \Theta_{z1} & \mu_{22} \Theta_{z2} \\ \chi_{11} m_1 & \chi_{12} m_2 & \kappa_{11} \Theta_{z1} & \kappa_{12} \Theta_{z2} \\ \chi_{21} m_1 & \chi_{22} m_2 & \kappa_{21} \Theta_{z1} & \kappa_{22} \Theta_{z2} \end{bmatrix}}_{\underline{\underline{M}}} \underbrace{\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{bmatrix}}_{\underline{\ddot{q}}} + \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\underline{\underline{E}}} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \varphi_1 \\ \varphi_2 \end{bmatrix}}_{\underline{q}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{0}}$$

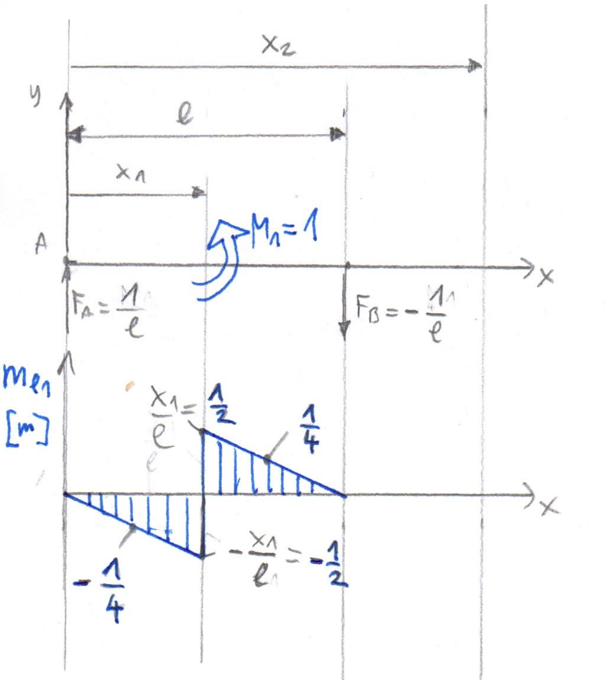
• a módosított tömegmátrix 2 részre bontásával:

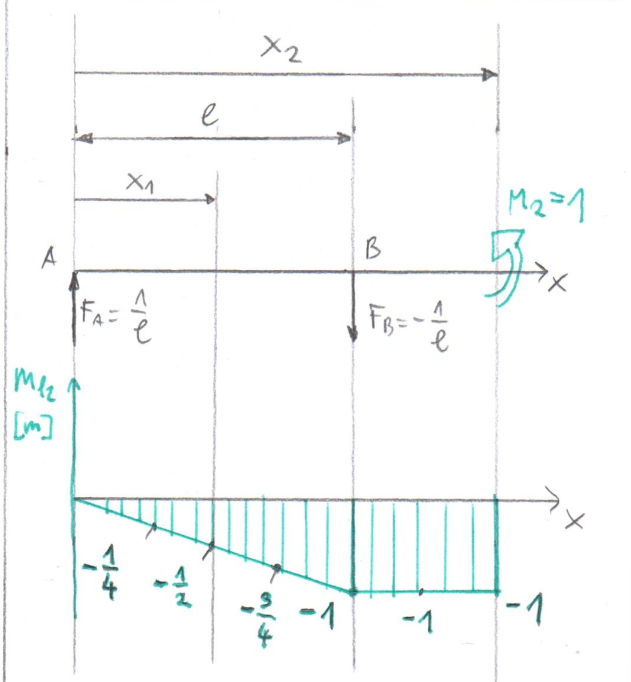
$$\underbrace{\begin{bmatrix} \delta_{11} & \delta_{12} & \mu_{11} & \mu_{12} \\ \delta_{21} & \delta_{22} & \mu_{21} & \mu_{22} \\ \chi_{11} & \chi_{12} & \kappa_{11} & \kappa_{12} \\ \chi_{21} & \chi_{22} & \kappa_{12} & \kappa_{22} \end{bmatrix}}_{\underline{\underline{D}}} \underbrace{\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & \Theta_{z1} & 0 \\ 0 & 0 & 0 & \Theta_{z2} \end{bmatrix}}_{\underline{\underline{M}}} \underbrace{\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{bmatrix}}_{\underline{\ddot{q}}} + \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\underline{\underline{E}}} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \varphi_1 \\ \varphi_2 \end{bmatrix}}_{\underline{q}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{0}}$$

• Maxwell-féle hatásszámok meghatározása Castigliano-tétellel









- a hajlító igénybevétel:

$$M_{hz} = F_1 m_{y_1} + F_2 m_{y_2} + M_1 m_{\varphi_1} + M_2 m_{\varphi_2}$$

- az alakváltozási energia:

$$U = \int_{(1{,}5\ell)} \frac{M_{hz}^2}{2I_z E}\,dx = \int_{(1{,}5\ell)} \frac{\left(F_1 m_{y_1} + F_2 m_{y_2} + M_1 m_{\varphi_1} + M_2 m_{\varphi_2}\right)^2}{2I_z E}\,dx$$

- a Castigliano tétel alapján az y_1 függőleges elmozdulás:

$$y_1 = \frac{\partial U}{\partial F_1} = \int_{(1{,}5\ell)} \frac{F_1 m_{y_1} + F_2 m_{y_2} + M_1 m_{\varphi_1} + M_2 m_{\varphi_2}}{I_z E} \cdot m_{y_1}\,dx =$$

$$= F_1 \underbrace{\int_{(1{,}5\ell)} \frac{m_{y_1}^2}{I_z E}\,dx}_{\delta_{11}} + F_2 \underbrace{\int_{(1{,}5\ell)} \frac{m_{y_2} m_{y_1}}{I_z E}}_{\delta_{12}} + M_1 \underbrace{\int_{(1{,}5\ell)} \frac{m_{\varphi_1} m_{y_1}}{I_z E}\,dx}_{\mu_{11}} + M_2 \underbrace{\int_{(1{,}5\ell)} \frac{m_{\varphi_2} m_{y_1}}{I_z E}\,dx}_{\mu_{12}}$$

- a Castigliano tétel alapján az y_2 függőleges elmozdulás

$$y_2 = \frac{\partial U}{\partial F_2} = \int_{(1{,}5\ell)} \frac{F_1 m_{y_1} + F_2 m_{y_2} + M_1 m_{\varphi_1} + M_2 m_{\varphi_2}}{I_z E}\, m_{y_2}\,dx =$$

$$= F_1 \underbrace{\int_{(1{,}5\ell)} \frac{m_{y_1} m_{y_2}}{I_z E}\,dx}_{\delta_{21}} + F_2 \underbrace{\int_{(1{,}5\ell)} \frac{m_{y_2}^2}{I_z E}\,dx}_{\delta_{22}} + M_1 \underbrace{\int_{(1{,}5\ell)} \frac{m_{\varphi_1} m_{y_2}}{I_z E}\,dx}_{\mu_{21}} + M_2 \underbrace{\int_{(1{,}5\ell)} \frac{m_{\varphi_2} m_{y_2}}{I_z E}\,dx}_{\mu_{22}}$$

- a Castigliano tétel alapján a φ_1 szögelfordulás:

$$\varphi_1 = \frac{\partial U}{\partial M_1} = \int_{(1{,}5\ell)} \frac{F_1 m_{y_1} + F_2 m_{y_2} + M_1 m_{\varphi_1} + M_2 m_{\varphi_2}}{I_z E}\, m_{\varphi_1}\,dx =$$

$$= F_1 \underbrace{\int_{(1{,}5\ell)} \frac{m_{y_1} m_{\varphi_1}}{I_z E}\,dx}_{\varkappa_{11}} + F_2 \underbrace{\int_{(1{,}5\ell)} \frac{m_{y_2} m_{\varphi_1}}{I_z E}\,dx}_{\varkappa_{12}} + M_1 \underbrace{\int_{(1{,}5\ell)} \frac{m_{\varphi_1}^2}{I_z E}\,dx}_{\varkappa_{11}} + M_2 \underbrace{\int_{(1{,}5\ell)} \frac{m_{\varphi_2} m_{\varphi_1}}{I_z E}\,dx}_{\kappa_{12}}$$

- a Castigliano tétel alapján a φ_2 szögelfordulás:

$$\varphi_2 = \frac{\partial U}{\partial M_2} = \int_{(1{,}5\ell)} \frac{F_1 m_{y_1} + F_2 m_{y_2} + M_1 m_{\varphi_1} + M_2 m_{\varphi_2}}{I_z E}\, m_{\varphi_2}\,dx =$$

$$= F_1 \underbrace{\int_{(1{,}5\ell)} \frac{m_{y_1} m_{\varphi_2}}{I_z E}\,dx}_{\varkappa_{21}} + F_2 \underbrace{\int_{(1{,}5\ell)} \frac{m_{y_2} m_{\varphi_2}}{I_z E}\,dx}_{\varkappa_{22}} + M_1 \underbrace{\int_{(1{,}5\ell)} \frac{m_{\varphi_1} m_{\varphi_2}}{I_z E}\,dx}_{\kappa_{21}} + M_2 \underbrace{\int_{(1{,}5\ell)} \frac{m_{\varphi_2}^2}{I_z E}\,dx}_{\kappa_{22}}$$

• A Maxwell-féle hatásszámok kiszámítása Simpsonnal:

$$\delta_{11} = \int_{(1{,}5\ell)} \frac{m_{y_1}^2}{I_z E}\,dx =$$

$$= \frac{1}{I_z E}\cdot\left[\frac{\frac{1}{2}\ell}{6}\cdot\left(0^2 + 4\cdot\left(\tfrac{1}{8}\ell\right)^2 + \left(\tfrac{1}{4}\ell\right)^2\right) + \frac{\frac{1}{2}\ell}{6}\cdot\left(\left(\tfrac{1}{4}\ell\right)^2 + 4\cdot\left(\tfrac{1}{8}\ell\right)^2 + 0^2\right)\right] =$$

$$= \frac{1}{I_z E}\cdot\left[\frac{\ell}{12}\cdot\left(\frac{4\ell^2}{64} + \frac{\ell^2}{16}\right) + \frac{\ell}{12}\cdot\left(\frac{\ell^2}{16} + \frac{4\ell^2}{64}\right)\right] = \frac{1}{I_z E}\cdot\left[\frac{\ell}{12}\cdot\frac{4\ell^2}{16}\right] = \frac{\ell^3}{48 I_z E}$$

$$\delta_{12} = \delta_{21} = \int_{(1{,}5\ell)} \frac{m_{y_1} m_{y_2}}{I_z E}\,dx =$$

$$= \frac{1}{I_z E}\cdot\left[\frac{\frac{1}{2}\ell}{6}\cdot\left(0\cdot 0 + 4\cdot\tfrac{1}{8}\ell\cdot\left(-\tfrac{1}{8}\ell\right) + \tfrac{1}{4}\ell\cdot\left(-\tfrac{1}{4}\ell\right)\right) + \frac{\frac{1}{2}\ell}{6}\cdot\left(\tfrac{1}{4}\ell\cdot\left(-\tfrac{1}{4}\ell\right) + 4\cdot\tfrac{1}{8}\ell\cdot\left(-\tfrac{3}{8}\ell\right) + 0\cdot\left(-\tfrac{1}{2}\ell\right)\right)\right] =$$

$$= \frac{1}{I_z E}\cdot\left[\frac{\ell}{12}\cdot\left(-\frac{4\ell^2}{64} - \frac{\ell^2}{16}\right) + \frac{\ell}{12}\cdot\left(-\frac{\ell^2}{16} - \frac{12\ell^2}{64}\right)\right] = \frac{1}{I_z E}\cdot\left[\frac{\ell}{12}\cdot\left(-\frac{6\ell^2}{16}\right)\right] = -\frac{\ell^3}{32 I_z E}$$

$$\mu_{11} = \chi_{11} = \int_{(1{,}5\ell)} \frac{m_{y_1} m_{e_1}}{I_z E}\,dx =$$

$$= \frac{1}{I_z E}\cdot\left[\frac{\frac{1}{2}\ell}{6}\cdot\left(0\cdot 0 + 4\cdot\tfrac{1}{8}\ell\cdot\left(-\tfrac{1}{4}\right) + \tfrac{1}{4}\ell\cdot\left(-\tfrac{1}{2}\right)\right) + \frac{\frac{1}{2}\ell}{6}\cdot\left(\tfrac{1}{4}\ell\cdot\tfrac{1}{2} + 4\cdot\tfrac{1}{8}\ell\cdot\tfrac{1}{4} + 0\cdot 0\right)\right] =$$

$$= \frac{1}{I_z E}\cdot\left[\frac{\ell}{12}\cdot\left(-\frac{\ell}{8} - \frac{\ell}{8}\right) + \frac{\ell}{12}\cdot\left(\frac{\ell}{8} + \frac{\ell}{8}\right)\right] = \frac{1}{I_z E}\cdot\left[\frac{\ell}{12}\cdot 0\right] = 0$$

$$\mu_{12} = \chi_{21} = \int_{(1{,}5\ell)} \frac{m_{y_1} m_{e_2}}{I_z E}\,dx =$$

$$= \frac{1}{I_z E}\cdot\left[\frac{\frac{1}{2}\ell}{6}\cdot\left(0\cdot 0 + 4\cdot\tfrac{1}{8}\ell\cdot\left(-\tfrac{1}{4}\right) + \tfrac{1}{4}\ell\cdot\left(-\tfrac{1}{2}\right)\right) + \frac{\frac{1}{2}\ell}{6}\cdot\left(\tfrac{1}{4}\ell\cdot\left(-\tfrac{1}{2}\right) + 4\cdot\tfrac{1}{8}\ell\cdot\left(-\tfrac{3}{4}\right) + 0\cdot(-1)\right)\right] =$$

$$= \frac{1}{I_z E}\cdot\left[\frac{\ell}{12}\cdot\left(-\frac{\ell}{8} - \frac{\ell}{8}\right) + \frac{\ell}{12}\cdot\left(-\frac{\ell}{8} - \frac{3\ell}{8}\right)\right] = \frac{1}{I_z E}\cdot\left[\frac{\ell}{12}\cdot\left(-\frac{6\ell}{8}\right)\right] = -\frac{\ell^2}{16 I_z E}$$

$$\delta_{22} = \int_{(1{,}5\ell)} \frac{m_{y_2}^2}{I_z E}\,dx =$$

$$= \frac{1}{I_z E}\cdot\left[\frac{\ell}{6}\cdot\left(0^2 + 4\cdot\left(-\tfrac{1}{4}\ell\right)^2 + \left(-\tfrac{1}{2}\ell\right)^2\right) + \frac{\frac{1}{2}\ell}{6}\cdot\left(\left(-\tfrac{1}{2}\ell\right)^2 + 4\cdot\left(-\tfrac{1}{4}\ell\right)^2 + 0^2\right)\right] =$$

$$= \frac{1}{I_z E}\cdot\left[\frac{\ell}{6}\cdot\left(\frac{4\ell^2}{16} + \frac{\ell^2}{4}\right) + \frac{\ell}{12}\left(\frac{\ell^2}{4} + \frac{4\ell^2}{16}\right)\right] = \frac{1}{I_z E}\cdot\left[\frac{2\ell^2}{4}\cdot\left(\frac{\ell}{6} + \frac{\ell}{12}\right)\right] = \frac{\ell^3}{8 I_z E}$$

$$\mu_{21} = \chi_{12} = \int\limits_{(1{,}5\ell)} \frac{M_{\eta_2} M_{\ell_1}}{I_z E}\, dx =$$

$$= \frac{1}{I_z E} \cdot \left[\frac{\frac{1}{2}\ell}{6} \cdot \left(0 \cdot 0 + 4 \cdot \left(-\frac{1}{8}\ell\right) \cdot \left(-\frac{1}{4}\right) + \left(-\frac{1}{4}\ell\right)\left(-\frac{1}{2}\right) \right) + \frac{\frac{1}{2}\ell}{6} \cdot \left(\left(-\frac{1}{4}\ell\right) \cdot \frac{1}{2} + 4 \cdot \left(-\frac{3}{8}\ell\right) \cdot \frac{1}{4} + \left(-\frac{1}{2}\ell\right) 0 \right) \right] =$$

$$= \frac{1}{I_z E} \cdot \left[\frac{\ell}{12} \cdot \left(\frac{\ell}{8} + \frac{\ell}{8} \right) + \frac{\ell}{12} \cdot \left(-\frac{\ell}{8} - \frac{3\ell}{8} \right) \right] = \frac{1}{I_z E} \cdot \left[\frac{\ell}{12} \cdot \left(-\frac{2\ell}{8} \right) \right] = -\frac{\ell^2}{48 I_z E}$$

$$\mu_{22} = \chi_{22} = \int\limits_{(1{,}5\ell)} \frac{M_{\eta_2} M_{\ell_2}}{I_z E}\, dx =$$

$$= \frac{1}{I_z E} \cdot \left[\frac{\ell}{6} \cdot \left(0 \cdot 0 + 4 \cdot \left(-\frac{1}{4}\ell\right) \cdot \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\ell\right) \cdot (-1) \right) + \frac{\frac{1}{2}\ell}{6} \cdot \left(\left(-\frac{1}{2}\ell\right) \cdot (-1) + 4 \cdot \left(-\frac{1}{4}\ell\right) \cdot (-1) + 0 \cdot (-1) \right) \right] =$$

$$= \frac{1}{I_z E} \cdot \left[\frac{\ell}{6} \cdot \left(\frac{\ell}{2} + \frac{\ell}{2} \right) + \frac{\ell}{12} \cdot \left(\frac{\ell}{2} + \ell \right) \right] = \frac{1}{I_z E} \cdot \left[\frac{\ell^2}{6} + \frac{\ell}{12} \cdot \frac{3\ell}{2} \right] = \frac{7\ell^2}{24 I_z E}$$

$$\kappa_{11} = \int\limits_{(1{,}5\ell)} \frac{M_{\ell_1}^2}{I_z E}\, dx =$$

$$= \frac{1}{I_z E} \cdot \left[\frac{\frac{1}{2}\ell}{6} \cdot \left(0^2 + 4 \cdot \left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{2}\right)^2 \right) + \frac{\frac{1}{2}\ell}{6} \cdot \left(\frac{1}{2}^2 + 4\left(\frac{1}{4}\right)^2 + 0^2 \right) \right] =$$

$$= \frac{1}{I_z E} \cdot \left[\frac{\ell}{12} \cdot \left(\frac{4}{16} + \frac{1}{4} \right) + \frac{\ell}{12} \cdot \left(\frac{1}{4} + \frac{4}{16} \right) \right] = \frac{1}{I_z E} \cdot \left[\frac{\ell}{12} \cdot \frac{1}{2} \cdot 2 \right] = \frac{\ell}{12 I_z E}$$

$$\kappa_{12} = \kappa_{21} = \int\limits_{(1{,}5\ell)} \frac{M_{\ell_1} M_{\ell_2}}{I_z E}\, dx =$$

$$= \frac{1}{I_z E} \cdot \left[\frac{\frac{1}{2}\ell}{6} \cdot \left(0 \cdot 0 + 4 \cdot \left(-\frac{1}{4}\right) \cdot \left(-\frac{1}{4}\right) + \left(-\frac{1}{2}\right) \cdot \left(-\frac{1}{2}\right) \right) + \frac{\frac{1}{2}\ell}{6} \cdot \left(\frac{1}{2} \cdot \left(-\frac{1}{2}\right) + 4 \cdot \frac{1}{4} \left(-\frac{3}{4}\right) + 0 \cdot (-1) \right) \right] =$$

$$= \frac{1}{I_z E} \cdot \left[\frac{\ell}{12} \cdot \left(\frac{1}{4} + \frac{1}{4} \right) + \frac{\ell}{12} \cdot \left(-\frac{1}{4} - \frac{3}{4} \right) \right] = \frac{1}{I_z E} \cdot \left[\frac{\ell}{12} \cdot \left(-\frac{2}{4} \right) \right] = -\frac{\ell}{24 I_z E}$$

$$\kappa_{22} = \int\limits_{(1{,}5\ell)} \frac{M_{\ell_2}^2}{I_z E}\, dx =$$

$$= \frac{1}{I_z E} \cdot \left[\frac{\ell}{6} \cdot \left(0^2 + 4 \cdot \left(-\frac{1}{2}\right)^2 + (-1)^2 \right) + \frac{\frac{1}{2}\ell}{6} \cdot \left((-1)^2 + 4 \cdot (-1)^2 + (-1)^2 \right) \right] =$$

$$= \frac{1}{I_z E} \cdot \left[\frac{\ell}{6} \cdot (1+1) + \frac{\ell}{12} \cdot (1+4+1) \right] = \frac{1}{I_z E} \cdot \left[\frac{2\ell}{6} + \frac{6\ell}{12} \right] = \frac{5\ell}{6 I_z E}$$

• A Maxwell-féle hatásszámok mátrixa:

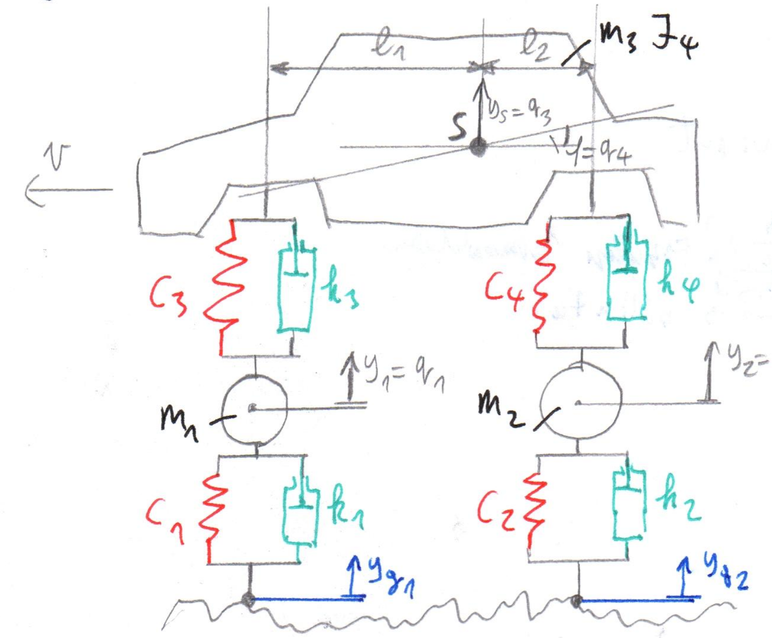
$$\underline{\underline{D}} = \begin{bmatrix} \delta_{11} & \delta_{12} & \mu_{11} & \mu_{12} \\ \delta_{21} & \delta_{22} & \mu_{21} & \mu_{22} \\ \chi_{11} & \chi_{12} & \kappa_{11} & \kappa_{12} \\ \chi_{21} & \chi_{22} & \kappa_{21} & \kappa_{22} \end{bmatrix} = \begin{bmatrix} \frac{l^3}{48 I_z E} & -\frac{l^3}{32 I_z E} & 0 & -\frac{l^2}{16 I_z E} \\ -\frac{l^3}{32 I_z E} & \frac{l^3}{8 I_z E} & -\frac{l^2}{48 I_z E} & \frac{7 l^2}{24 I_z E} \\ 0 & -\frac{l^2}{48 I_z E} & \frac{l}{12 I_z E} & -\frac{l}{24 I_z E} \\ -\frac{l^2}{16 I_z E} & \frac{7 l^2}{24 I_z E} & -\frac{l}{24 I_z E} & \frac{5 l}{6 I_z E} \end{bmatrix}$$

• A mozgásegyenlet:

$$\underbrace{\begin{bmatrix} \frac{l^3}{48 I_z E} & -\frac{l^3}{32 I_z E} & 0 & -\frac{l^2}{16 I_z E} \\ -\frac{l^3}{32 I_z E} & \frac{l^3}{8 I_z E} & -\frac{l^2}{48 I_z E} & \frac{7 l^2}{24 I_z E} \\ 0 & -\frac{l^2}{48 I_z E} & \frac{l}{12 I_z E} & -\frac{l}{24 I_z E} \\ -\frac{l^2}{16 I_z E} & \frac{7 l^2}{24 I_z E} & -\frac{l}{24 I_z E} & \frac{5 l}{6 I_z E} \end{bmatrix}}_{\underline{\underline{D}}} \underbrace{\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & \Theta_{z1} & 0 \\ 0 & 0 & 0 & \Theta_{z2} \end{bmatrix}}_{\underline{\underline{M}}} \underbrace{\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{bmatrix}}_{\underline{\ddot{q}}} +$$

$$+ \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\underline{\underline{E}}} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \varphi_1 \\ \varphi_2 \end{bmatrix}}_{\underline{q}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{0}}$$

FÉL JÁRMŰMODELL



• adott:

l_1, l_2

m_1, m_2, m_3, J_4

c_1, c_2, c_3, c_4

k_1, k_2, k_3, k_4

y_{g1}, y_{g2}

• ált. koordináták:

$q_1 = y_1$

$q_2 = y_2$

$q_3 = y_S$

$q_4 = \varphi$

• Lagrange-féle másodfajú mozgásegyenlet:

$$\frac{d}{dt}\left[\frac{\partial E}{\partial \dot q_i}\right] - \frac{\partial E}{\partial q_i} = Q_{ci} + Q_{ki} + Q_{yi} \qquad i = 1,2,3,4$$

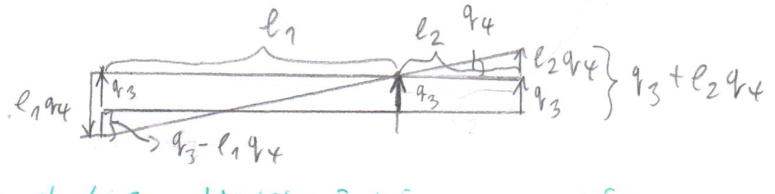
① A rendszer teljes kinetikai energiája:

$$E = \frac{1}{2} m_1 \dot q_1^2 + \frac{1}{2} m_2 \dot q_2^2 + \frac{1}{2} m_3 \dot q_3^2 + \frac{1}{2} J_4 \dot q_4^2$$

② A rendszer teljes rugóenergiája:

$$V = V_1 + V_2 + V_3 + V_4 = \frac{1}{2}\frac{f_1^2}{c_1} + \frac{1}{2}\frac{f_2^2}{c_2} + \frac{1}{2}\frac{f_3^2}{c_3} + \frac{1}{2}\frac{f_4^2}{c_4} =$$

$$= \frac{1}{2}\frac{(q_1 - y_{g1})^2}{c_1} + \frac{1}{2}\frac{(q_2 - y_{g2})^2}{c_2} + \frac{1}{2}\frac{(q_3 - l_1 q_4 - q_1)^2}{c_3} + \frac{1}{2}\frac{(q_3 + l_2 q_4 - q_2)^2}{c_4}$$



③ A rendszer teljes disszipációs energiája:

$$D = D_1 + D_2 + D_3 + D_4 = \frac{1}{2} k_1 (\dot q_1 - \dot y_{g1})^2 + \frac{1}{2} k_2 (\dot q_2 - \dot y_{g2})^2 +$$

$$+ \frac{1}{2} k_3 (\dot q_3 - l_1 \dot q_4 - \dot q_1)^2 + \frac{1}{2} k_4 (\dot q_3 + l_2 \dot q_4 - \dot q_2)^2$$

• Az 1. mozgásegyenlet: $\boxed{i=1}$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_1}\right)}_{1} - \underbrace{\frac{\partial E}{\partial q_1}}_{=0} = \underbrace{Q_{c1}}_{2} + \underbrace{Q_{k1}}_{3} + Q_{y1}$$

$$Q_{y1} = \underbrace{Q_{yc1}}_{2} + \underbrace{Q_{yk1}}_{3}$$

1 $\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_1}\right) = m_1 \ddot{q}_1$

2 $Q_{c1} + Q_{yc1} = -\frac{\partial U}{\partial q_1} = -\left(\frac{q_1 - y_{q1}}{c_1} + \frac{q_3 - \ell_1 q_4 - q_1}{c_3}(-1)\right) =$

$= -\frac{1}{c_1} q_1 + \frac{1}{c_1} y_{y1} + \frac{1}{c_3} q_3 - \frac{\ell_1}{c_3} q_4 - \frac{1}{c_3} q_1 =$

$= \underbrace{-\left(\frac{1}{c_1} + \frac{1}{c_3}\right) q_1 + \frac{1}{c_3} q_3 - \frac{\ell_1}{c_3} q_4}_{Q_{c1}} + \underbrace{\frac{1}{c_1} y_{q1}}_{Q_{yc1}}$

3 $Q_{k1} + Q_{yk1} = -\frac{\partial D}{\partial \dot{q}_1} = -\left(k_1(\dot{q}_1 - \dot{y}_{y1}) + k_3(\dot{q}_3 - \ell_1 \dot{q}_4 - \dot{q}_1)(-1)\right) =$

$= -k_1 \dot{q}_1 + k_1 \dot{y}_{y1} + k_3 \dot{q}_3 - k_3 \ell_1 \dot{q}_4 - k_3 \dot{q}_1 =$

$= \underbrace{-(k_1 + k_3)\dot{q}_1 + k_3 \dot{q}_3 - k_3 \ell_1 \dot{q}_4}_{Q_{k1}} + \underbrace{k_1 \dot{y}_{y1}}_{Q_{yk1}}$

'A 2. mozgásegyenlet: $\boxed{i=2}$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial E}{\partial \dot q_2}\right)}_{1} - \underbrace{\frac{\partial E}{\partial q_2}}_{=0} = \underbrace{Q_{c2}}_{2} + \underbrace{Q_{k2}}_{3} + Q_{q2}$$

$$Q_{g_2} = \underbrace{Q_{g_{c2}}}_{2} + \underbrace{Q_{y_{k2}}}_{3}$$

1 $$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot q_2}\right) = m_2 \ddot q_2$$

2 $$Q_{c2} + Q_{y_{c2}} = -\frac{\partial U}{\partial q_2} = -\left(\frac{q_2 - y_{q2}}{c_2} + \frac{q_3 + \ell_2 q_4 - q_2}{c_4}(-1)\right) =$$

$$= -\frac{1}{c_2} q_2 + \frac{1}{c_2} y_{q2} + \frac{1}{c_4} q_3 + \frac{\ell_2}{c_4} q_4 - \frac{1}{c_4} q_2 =$$

$$= \underbrace{-\left(\frac{1}{c_2} + \frac{1}{c_4}\right) q_2 + \frac{1}{c_4} q_3 + \frac{\ell_2}{c_4} q_4}_{Q_{c2}} + \underbrace{\frac{1}{c_2} y_{q2}}_{Q_{y_{c2}}}$$

3 $$Q_{k2} + Q_{y_{k2}} = -\frac{\partial D}{\partial \dot q_2} = -\left(k_2(\dot q_2 - \dot y_{q2}) + k_4(\dot q_3 + \ell_2 \dot q_4 - \dot q_2)(-1)\right) =$$

$$= -k_2 \dot q_2 + k_2 \dot y_{q2} + k_4 \dot q_3 + k_4 \ell_2 \dot q_4 - k_4 \dot q_2 =$$

$$= \underbrace{-(k_2 + k_4)\dot q_2 + k_4 \dot q_3 + k_4 \ell_2 \dot q_4}_{Q_{k2}} + \underbrace{k_2 \dot y_{q2}}_{Q_{y_{k2}}}$$

• A 3. mozgásegyenlet : $\boxed{i=3}$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_3}\right)}_{1} - \underbrace{\frac{\partial E}{\partial q_3}}_{=0} = \underbrace{Q_{c3}}_{2} + \underbrace{Q_{k3}}_{3} + Q_{q3}$$

$$Q_{q3} = \underbrace{Q_{qc3}}_{2} + \underbrace{Q_{qk3}}_{3}$$

1 $$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_3}\right) = m_3 \ddot{q}_3$$

2 $$Q_{c3} + Q_{qc3} = -\frac{\partial U}{\partial q_3} = -\left(\frac{q_3 - \ell_1 q_4 - q_1}{c_3} + \frac{q_3 + \ell_2 q_4 - q_2}{c_4}\right) =$$

$$= -\frac{1}{c_3} q_3 + \frac{\ell_1}{c_3} q_4 + \frac{1}{c_3} q_1 - \frac{1}{c_4} q_3 - \frac{\ell_2}{c_4} q_4 + \frac{1}{c_4} q_2 =$$

$$= \underbrace{\frac{1}{c_3} q_1 + \frac{1}{c_4} q_2 - \left(\frac{1}{c_3} + \frac{1}{c_4}\right) q_3 + \left(\frac{\ell_1}{c_3} - \frac{\ell_2}{c_4}\right) q_4}_{Q_{c3}}$$

3 $$Q_{k3} + Q_{qk3} = -\frac{\partial D}{\partial \dot{q}_3} = -\left(k_3(\dot{q}_3 - \ell_1 \dot{q}_4 - \dot{q}_1) + k_4(\dot{q}_3 + \ell_2 \dot{q}_4 - \dot{q}_2)\right) =$$

$$= -k_3 \dot{q}_3 + k_3 \ell_1 \dot{q}_4 + k_3 \dot{q}_1 - k_4 \dot{q}_3 - k_4 \ell_2 \dot{q}_4 + k_4 \dot{q}_2 =$$

$$= \underbrace{k_3 \dot{q}_1 + k_4 \dot{q}_2 - (k_3 + k_4) \dot{q}_3 + (k_3 \ell_1 - k_4 \ell_2) \dot{q}_4}_{Q_{k3}}$$

• A 4. mozgásegyenlet : $\boxed{i=4}$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial E}{\partial \dot q_4}\right)}_{1} - \underbrace{\frac{\partial E}{\partial q_4}}_{=0} = \underbrace{Q_{c4}}_{2} + \underbrace{Q_{k4}}_{3} + Q_{g4}$$

$$Q_{g4} = \underbrace{Q_{g_{c4}}}_{2} + \underbrace{Q_{g_{k4}}}_{3}$$

$$1 \quad \frac{d}{dt}\left(\frac{\partial E}{\partial \dot q_4}\right) = J_4 \ddot q_4$$

$$2 \quad Q_{c4} + Q_{g_{c4}} = -\frac{\partial U}{\partial q_4} = -\left(\frac{q_3 - l_1 q_4 - q_1}{c_3}(-l_1) + \frac{q_3 + l_2 q_4 - q_2}{c_4} l_2\right) =$$

$$= +\frac{l_1}{c_3} q_3 - \frac{l_1^2}{c_3} q_4 - \frac{l_1}{c_3} q_1 - \frac{l_2}{c_4} q_3 - \frac{l_2^2}{c_4} q_4 + \frac{l_2}{c_4} q_2 =$$

$$= \underbrace{-\frac{l_1}{c_3} q_1 + \frac{l_2}{c_4} q_2 + \left(\frac{l_1}{c_3} - \frac{l_2}{c_4}\right) q_3 - \left(\frac{l_1^2}{c_3} + \frac{l_2^2}{c_4}\right) q_4}_{Q_{c4}}$$

$$3 \quad Q_{k4} + Q_{g_{k4}} = -\frac{\partial D}{\partial \dot q_4} = -\left(k_3(\dot q_3 - l_1 \dot q_4 - \dot q_1)(-l_1) + k_4(\dot q_3 + l_2 \dot q_4 - \dot q_2) l_2\right) =$$

$$= k_3 l_1 \dot q_3 - k_3 l_1^2 \dot q_4 - k_3 l_1 \dot q_1 - k_4 l_2 \dot q_3 - k_4 l_2^2 \dot q_4 + k_4 l_2 \dot q_2 =$$

$$= \underbrace{-k_3 l_1 \dot q_1 + k_4 l_2 \dot q_2 + (k_3 l_1 - k_4 l_2)\dot q_3 - (k_3 l_1^2 + k_4 l_2^2)\dot q_4}_{Q_{k4}}$$

• Visszahelyettesítve, átrendezve a mozgásegyenlet rendszer:

$$i=1 \Rightarrow m_1\ddot{q}_1 + (k_1+k_3)\dot{q}_1 - k_3\dot{q}_3 + k_3 l_1 \dot{q}_4 + \left(\frac{1}{c_1}+\frac{1}{c_3}\right)q_1 - \frac{1}{c_3}q_3 + \frac{l_1}{c_3}q_4 = k_1\dot{y}_{g1} + \frac{1}{c_1}y_{g1}$$

$$i=2 \Rightarrow m_2\ddot{q}_2 + (k_2+k_4)\dot{q}_2 - k_4\dot{q}_3 - k_4 l_2 \dot{q}_4 + \left(\frac{1}{c_2}+\frac{1}{c_4}\right)q_2 - \frac{1}{c_4}q_3 - \frac{l_2}{c_4}q_4 = k_2\dot{y}_{g2} + \frac{1}{c_2}y_{g2}$$

$$i=3 \Rightarrow m_3\ddot{q}_3 - k_3\dot{q}_1 - k_4\dot{q}_2 + (k_3+k_4)\dot{q}_3 - (k_3 l_1 - k_4 l_2)\dot{q}_4 - \frac{1}{c_3}q_1 - \frac{1}{c_4}q_2 + \left(\frac{1}{c_3}+\frac{1}{c_4}\right)q_3 - \left(\frac{l_1}{c_3}-\frac{l_2}{c_4}\right)q_4 = 0$$

$$i=4 \Rightarrow J_4\ddot{q}_4 + k_3 l_1\dot{q}_1 - k_4 l_2\dot{q}_2 - (k_3 l_1 - k_4 l_2)\dot{q}_3 + (k_3 l_1^2 + k_4 l_2^2)\dot{q}_4 + \frac{l_1}{c_3}q_1 - \frac{l_2}{c_4}q_2 - \left(\frac{l_1}{c_3}-\frac{l_2}{c_4}\right)q_3 + \left(\frac{l_1^2}{c_3}+\frac{l_2^2}{c_4}\right)q_4 = 0$$

• átírva mátrixos alakra:

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & J_4 \end{bmatrix}}_{\underline{\underline{M}}}\underbrace{\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \end{bmatrix}}_{\underline{\ddot{q}}} + \underbrace{\begin{bmatrix} k_1+k_3 & 0 & -k_3 & k_3 l_1 \\ 0 & k_2+k_4 & -k_4 & -k_4 l_2 \\ -k_3 & -k_4 & k_3+k_4 & -(k_3 l_1 - k_4 l_2) \\ k_3 l_1 & -k_4 l_2 & -(k_3 l_1 - k_4 l_2) & k_3 l_1^2 + k_4 l_2^2 \end{bmatrix}}_{\underline{\underline{D}}}\underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}}_{\underline{\dot{q}}} + \underbrace{\begin{bmatrix} \frac{1}{c_1}+\frac{1}{c_3} & 0 & -\frac{1}{c_3} & \frac{l_1}{c_3} \\ 0 & \frac{1}{c_2}+\frac{1}{c_4} & -\frac{1}{c_4} & -\frac{l_2}{c_4} \\ -\frac{1}{c_3} & -\frac{1}{c_4} & \frac{1}{c_3}+\frac{1}{c_4} & -\left(\frac{l_1}{c_3}-\frac{l_2}{c_4}\right) \\ \frac{l_1}{c_3} & -\frac{l_2}{c_4} & -\left(\frac{l_1}{c_3}-\frac{l_2}{c_4}\right) & \frac{l_1^2}{c_3}+\frac{l_2^2}{c_4} \end{bmatrix}}_{\underline{\underline{K}}}\underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}}_{\underline{q}} = \overbrace{\underbrace{\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\underline{\underline{W}}}\underbrace{\begin{bmatrix} \dot{y}_{g1} \\ \dot{y}_{g2} \end{bmatrix}}_{\underline{\dot{y}}_g} + \underbrace{\begin{bmatrix} \frac{1}{c_1} & 0 \\ 0 & \frac{1}{c_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\underline{\underline{V}}}\underbrace{\begin{bmatrix} y_{g1} \\ y_{g2} \end{bmatrix}}_{\underline{y}_g}}^{\underline{Q}_g}$$

$$\underline{\underline{M}}\,\underline{\ddot{q}} + \underline{\underline{D}}\,\underline{\dot{q}} + \underline{\underline{K}}\,\underline{q} = \underbrace{\underline{\underline{W}}\,\underline{\dot{y}}_g + \underline{\underline{V}}\,\underline{y}_g}_{\underline{Q}_g}$$