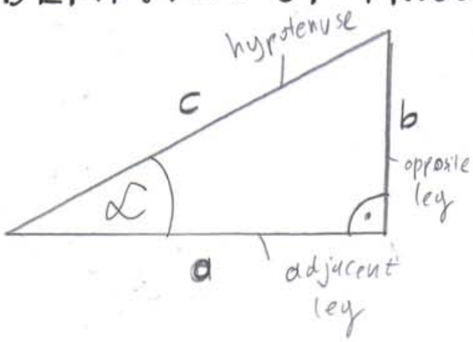
MATHEMATICAL BASES

1. TRIGONOMETRY

1.1 DEFINITION OF TRIGONOMETRIC FUNCTIONS



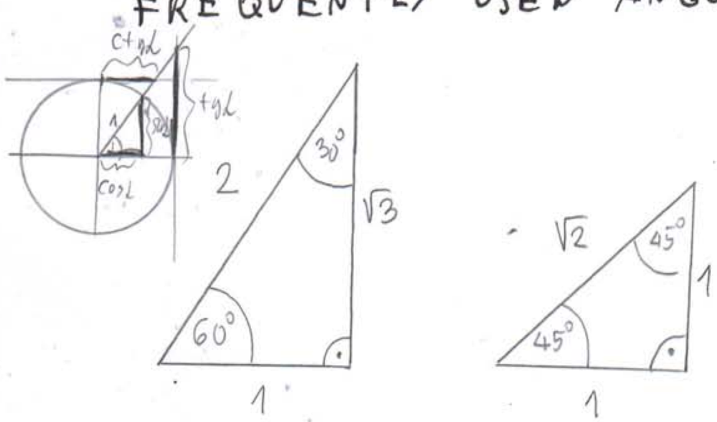
$$\sin\alpha = \frac{b}{c} \Rightarrow b = c\cdot\sin\alpha$$

$$\cos\alpha = \frac{a}{c} \Rightarrow a = c\cdot\cos\alpha$$

$$tg\,\alpha = \frac{b}{a}$$

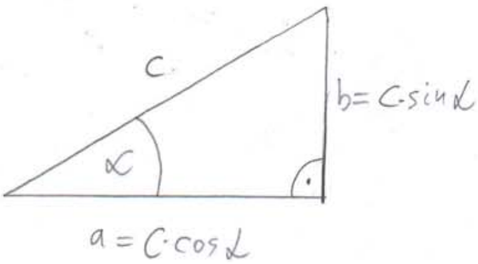
$$ctg\,\alpha = \frac{a}{b} = \frac{1}{tg\,\alpha}$$

1.2 VALUES OF TRIGONOMETRIC FUNCTIONS OF FREQUENTLY USED ANGLES



α	$\sin\alpha$	$\cos\alpha$	$tg\,\alpha$	$ctg\,\alpha$
0°	$0 = \frac{\sqrt{0}}{2}$	$1 = \frac{\sqrt{4}}{2}$	0	$\to\infty$
30°	$\frac{1}{2} = \frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
45°	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	1	1
60°	$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$	$\frac{1}{2} = \frac{\sqrt{1}}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
90°	$1 = \frac{\sqrt{4}}{2}$	$0 = \frac{\sqrt{0}}{2}$	$\to\infty$	0

1.3 PYTHAGOREAN THEOREM



$$a^2 + b^2 = c^2$$

$$\cancel{c^2}\cos^2\alpha + \cancel{c^2}\sin^2\alpha = \cancel{c^2}$$

$$\cos^2\alpha + \sin^2\alpha = 1$$

2. VECTOR ALGEBRA

• **vector**: geometric object, that has • magnitude • direction ⇒ It is a directed line segment

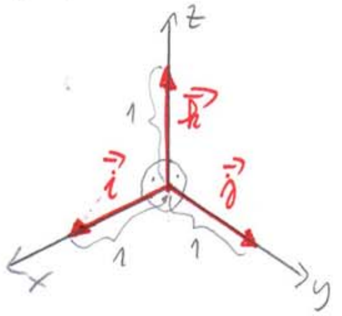
notation: $\vec{a}$

2.1 DEFINE A VECTOR

→ Take 3 mutually perpendicular base vectors in space pointing in the x, y and z directions

⇓

Using these base vectors, any vector in space can be uniqely obtained as a linear combination of unit vectors

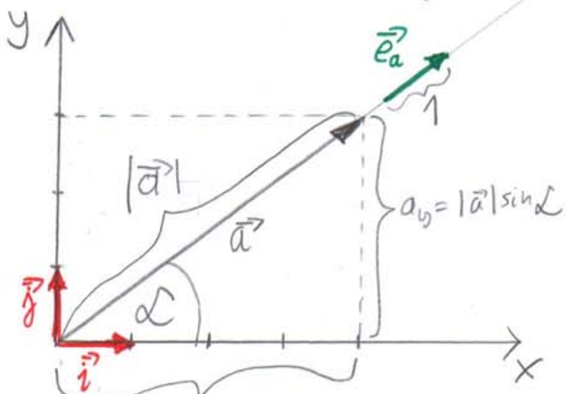


base vectors: $\vec{i}, \vec{j}, \vec{k}$

$|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$

$\vec{i} \perp \vec{j}$; $\vec{i} \perp \vec{k}$; $\vec{j} \perp \vec{k}$

• Consider an arbitrary vector in xy plane:



• $\vec{a} = \underbrace{a_x}_{\text{coordinate}} \vec{i} + \underbrace{a_y}_{\text{component}} \vec{j}$

• length of $\vec{a}$ ⇒ Pythagorean theorem:

$$\boxed{|\vec{a}| = \sqrt{a_x^2 + a_y^2}} \qquad \alpha = \operatorname{atan} \frac{a_y}{a_x}$$

• with $|\vec{a}|$ and α:

$$\vec{a} = \underbrace{|\vec{a}| \cos\alpha}_{a_x} \vec{i} + \underbrace{|\vec{a}| \sin\alpha}_{a_y} \vec{j} =$$

$$= |\vec{a}| \underbrace{(\cos\alpha\, \vec{i} + \sin\alpha\, \vec{j})}_{\vec{e_a}}$$

$$\boxed{\vec{a} = |\vec{a}|\, \vec{e_a}}$$

$|\vec{a}|$: magnitude (length) of $\vec{a}$

$\vec{e_a}$: unit direction vector of $\vec{a}$

Determining $\vec{e_a}$ unit direction vector

a) if we know

$\vec{a} = a_x \vec{i} + a_y \vec{j}$

⇓

$$\boxed{\vec{e_a} = \frac{\vec{a}}{|\vec{a}|} = \frac{a_x \vec{i} + a_y \vec{j}}{\sqrt{a_x^2 + a_y^2}}}$$

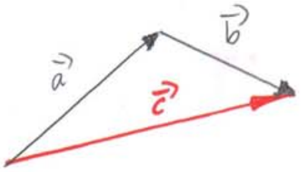
b) if we know:

α

⇓

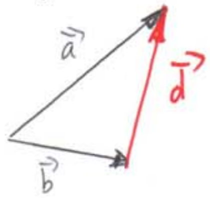
$$\boxed{\vec{e_a} = \cos\alpha\, \vec{i} + \sin\alpha\, \vec{j}}$$

2.2 ADDITION OF VECTORS



$$\vec{a}+\vec{b}=(a_x\vec{i}+a_y\vec{j})+(b_x\vec{i}+b_y\vec{j})=\underbrace{(a_x+b_x)}_{c_x}\vec{i}+\underbrace{(a_y+b_y)}_{c_y}\vec{j}=\vec{c}$$

2.3 SUBTRACTION OF VECTORS

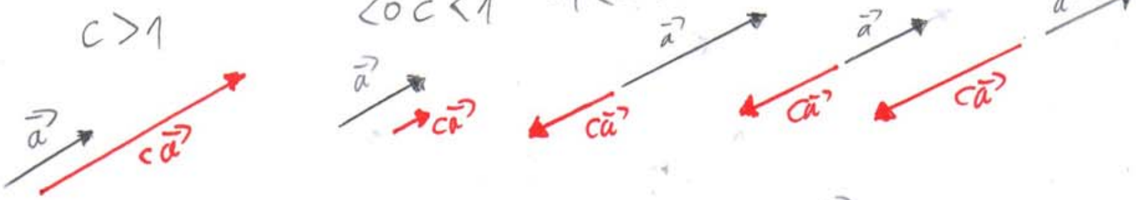


$$\vec{a}-\vec{b}=(a_x\vec{i}+a_y\vec{j})-(b_x\vec{i}+b_y\vec{j})=\underbrace{(a_x-b_x)}_{d_x}\vec{i}+\underbrace{(a_y-b_y)}_{d_y}\vec{j}=\vec{d}$$

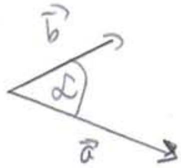
2.4 MULTIPLICATION WITH SCALAR

$$c\vec{a}=c(a_x\vec{i}+a_y\vec{j})=ca_x\vec{i}+ca_y\vec{j}$$

$c>1$ $\quad 0<c<1$ $\quad -1<c<0$ $\quad c=-1$ $\quad c<-1$



2.5 DOT PRODUCT



- result: scalar
- interpretation: $\boxed{\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\alpha}$

$$\Downarrow$$

$$\boxed{\vec{a}\perp\vec{b}\iff\vec{a}\cdot\vec{b}=0}$$

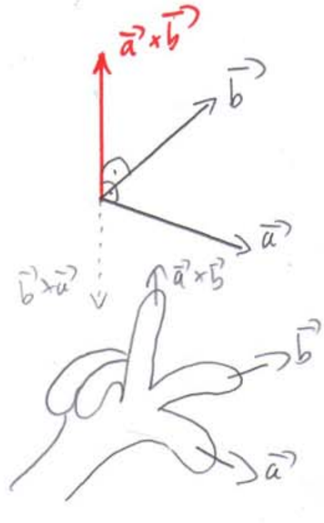
- dot product of base vectors: $\vec{i}\cdot\vec{i}=1\quad\vec{j}\cdot\vec{j}=1\quad\vec{k}\cdot\vec{k}=1$
  $\vec{i}\cdot\vec{j}=0\quad\vec{i}\cdot\vec{k}=0\quad\vec{j}\cdot\vec{k}=0$
- calculation with base vectors:

$$\boxed{\vec{a}\cdot\vec{b}}=(a_x\vec{i}+a_y\vec{j}+a_z\vec{k})\cdot(b_x\vec{i}+b_y\vec{j}+b_z\vec{k})=\boxed{a_xb_x+a_yb_y+a_zb_z}$$

- application:
  - → checking perpendicularity
  - → calculate angle between 2 vectors: $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\alpha\Rightarrow\cos\alpha=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$
  - → coordinate in given direction

$$\boxed{\begin{aligned}a_{\parallel}&=\vec{e}\cdot\vec{a}\\ \vec{a}_{\parallel}&=a_{\parallel}\vec{e}=(\vec{e}\cdot\vec{a})\,\vec{e}\end{aligned}}$$

2.6 CROSS PRODUCT



- result: vector
- interpretation:
  - → magnitude: $\boxed{|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\alpha}$
  - → direction:
    - $\vec{a}\times\vec{b} \perp a$
    - $\vec{a}\times b \perp b$
    - direction is given by the right-hand rule

$$\boxed{\vec{a}\parallel\vec{b} \iff \vec{a}\times\vec{b} = \vec{0}}$$

- cross product of base vectors:

$\vec{i}\times\vec{j} = \vec{k}$ $\quad\vec{j}\times\vec{k} = \vec{i}$ $\quad\vec{k}\times\vec{i} = \vec{j}$

$\vec{j}\times\vec{i} = -\vec{k}$ $\quad\vec{k}\times\vec{j} = -\vec{i}$ $\quad\vec{i}\times\vec{k} = -\vec{j}$

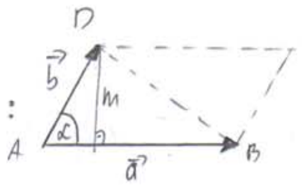
- calculation with base vectors:
  - → <u>method 1:</u> $\vec{a}\times\vec{b} = (a_x\vec{i} + a_y\vec{j})\times(b_x\vec{i} + b_y\vec{j}) =$

$$= a_xb_x\underbrace{\vec{i}\times\vec{i}}_{\vec{0}} + a_xb_y\underbrace{\vec{i}\times\vec{j}}_{\vec{k}} + a_yb_x\underbrace{\vec{j}\times\vec{i}}_{-\vec{k}} + a_yb_y\underbrace{\vec{j}\times\vec{j}}_{\vec{0}} = (a_xb_y - a_yb_x)\,\vec{k}$$

  - → <u>method 2:</u>

$$\vec{a}\times\vec{b} = \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \vec{i}\,(a_yb_z - a_zb_y) - \vec{j}\,(a_xb_z - a_zb_x) + \\ + \vec{k}\,(a_xb_y - a_yb_x)$$

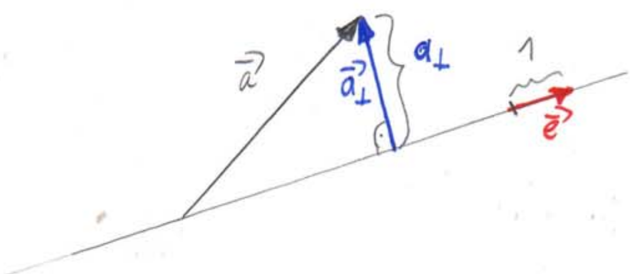
- application:
  - → checking parallelism
  - → area of parallelogram / triangle:



$T_{ABCD} = |\vec{a}\times\vec{b}|$

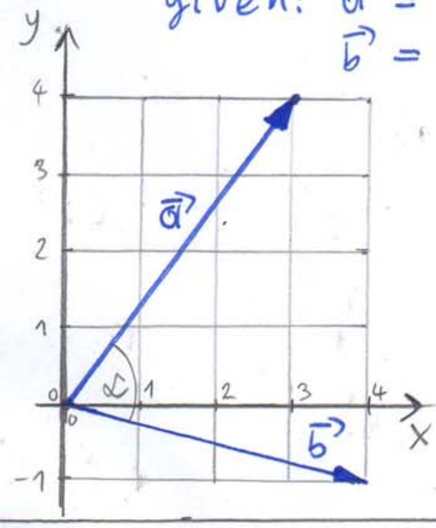
$T_{ABD} = \dfrac{|\vec{a}\times\vec{b}|}{2}$

  - → coordinate perpendicular to given direction



$$\boxed{\begin{aligned} a_\perp &= |\vec{e}\times\vec{a}| \\ \vec{a}_\perp &= (\vec{e}\times\vec{a})\times\vec{e} \end{aligned}}$$

EXERCISE 1.1



given: $\vec{a} = 3\vec{i} + 4\vec{j}$

$\vec{b} = 4i - 1\vec{j}$

task:

a) $\vec{a} + \vec{b}$
b) $\vec{a} - \vec{b}$
c) $\vec{b} - \vec{a}$
d) $0.5\,\vec{a}$
e) $-2\vec{b}$
f) $|\vec{a}|$
g) $|\vec{b}|$
h) $\vec{e}_a$
i) $\vec{e}_b$
j) $\vec{a} \cdot \vec{b}$
k) α
l) $\vec{a} \times \vec{b}$
m) $\vec{b} \times \vec{a}$
n) $a_{\parallel b}$
o) $\vec{a}_{\parallel b}$
p) $a_{\perp b}$
q) $\vec{a}_{\perp b}$

a) $\vec{a} + \vec{b} = (3\vec{i} + 4\vec{j}) + (4\vec{i} - 1\vec{j}) = (3+4)\,\vec{i} + (4-1)\,\vec{j} = 7\vec{i} + 3\vec{j}$

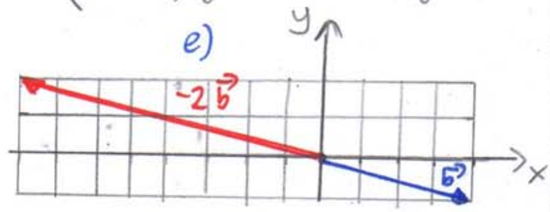
b) $\vec{a} - \vec{b} = (3\vec{i} + 4\vec{j}) - (4\vec{i} - 1\vec{j}) = (3-4)\,\vec{i} + (4-(-1))\,\vec{j} = -1\vec{i} + 5\vec{j}$

c) $\vec{b} - \vec{a} = (4\vec{i} - 1\vec{j}) - (3\vec{i} + 4\vec{j}) = (4-3)\,\vec{i} + (-1-4)\,\vec{j} = 1\vec{i} - 5\vec{j}$

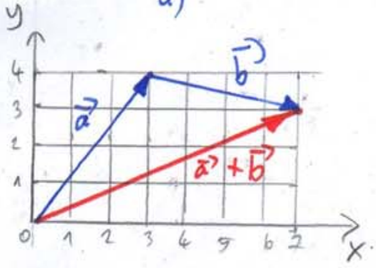
d) $0.5\,\vec{a} = 0.5\,(3\vec{i} + 4\vec{j}) = 1{,}5\,\vec{i} + 2\vec{j}$

e) $-2\vec{b} = -2\,(4\vec{i} - 1\vec{j}) = -8\vec{i} + 2\vec{j}$

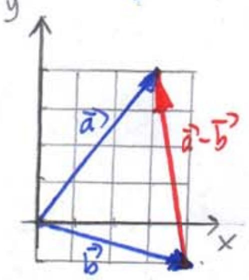
e)



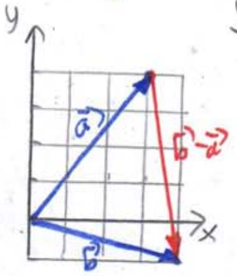
a)



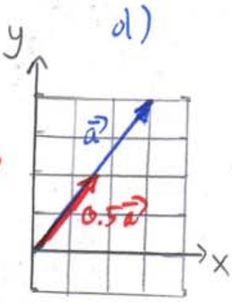
b)



c)



d)



f) $|\vec{a}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

g) $|\vec{b}| = \sqrt{4^2 + (-1)^2} = \sqrt{17} = 4{,}12$

h) $\vec{e}_a = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{3\vec{i} + 4\vec{j}}{5} = \dfrac{3}{5}\vec{i} + \dfrac{4}{5}\vec{j} = 0.6\,\vec{i} + 0.8\,\vec{j}$

i) $\vec{e}_b = \dfrac{\vec{b}}{|\vec{b}|} = \dfrac{4\vec{i} - 1\vec{j}}{4{,}12} = 0.97\,\vec{i} - 0.24\,\vec{j}$

j) $\vec{a}\cdot\vec{b} = (3\vec{i}+4\vec{j})\cdot(4\vec{i}-1\vec{j}) = 3\cdot 4 + 4\cdot(-1) = 12-4 = 8$

k) $\cos\alpha = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \dfrac{8}{5\cdot 4{,}12} = 0.388 \Rightarrow \alpha = 67{,}17^\circ$

l) $\vec{a}\times\vec{b} = (3\vec{i}+4\vec{j})\times(4\vec{i}-1\vec{j}) = 3\cdot 4\,\overbrace{\vec{i}\times\vec{i}}^{\vec{0}} + \overbrace{3\cdot(-1)}^{-3}\,\overbrace{\vec{i}\times\vec{j}}^{\vec{k}} +$

$+\underbrace{4\cdot 4}_{16}\,\underbrace{\vec{j}\times\vec{i}}_{-\vec{k}} + 4(-1)\underbrace{\vec{j}\times\vec{j}}_{\vec{0}} = -3\vec{k} - 16\vec{k} = -19\vec{k}$

$$\vec{a}\times\vec{b} = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 0 \\ 4 & -1 & 0 \end{vmatrix} \end{array} = \vec{i}\overbrace{(4\cdot 0-(-1)\cdot 0)}^{0} - \vec{j}\overbrace{(3\cdot 0-4\cdot 0)}^{0} + \vec{k}\overbrace{(3\cdot(-1)-4\cdot 4)}^{-19} =$$

$$= -19\vec{k}$$

$|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\alpha = 5\cdot 4{,}12\cdot\sin 67{,}17^\circ = 19$

m) $\vec{b}\times\vec{a} = -\vec{a}\times\vec{b} = 19\vec{k}$

n) $a_{\parallel b} = \vec{a}\cdot\vec{e_b} = (3\vec{i}+4\vec{j})\cdot(0.97\vec{i}-0.24\vec{j}) =$

$= 3\cdot 0{,}97 + 4\cdot(-0{,}24) = 1{,}95$

o) $\vec{a}_{\parallel b} = a_\parallel\vec{e_b} = 1.95\,(0.97\vec{i}-0{,}24\vec{j}) =$

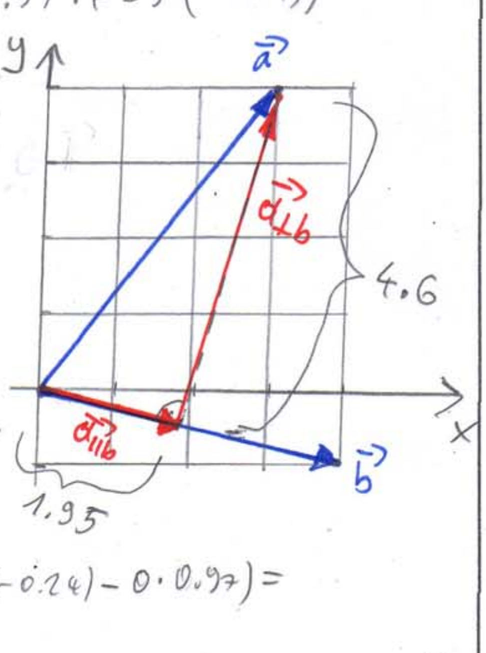
$= 1.89\vec{i} - 0.47\vec{j}$

p) $$\vec{e_b}\times\vec{a} = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.97 & -0.24 & 0 \\ 3 & 4 & 0 \end{vmatrix} \end{array} = \vec{i}\cdot 0 - \vec{j}\cdot 0 + \vec{k}\,(0.97\cdot 4 - 3\cdot(-0.24)) =$$

$$= 4.6\,\vec{k}$$

or: $\vec{e_b}\times\vec{a} = \dfrac{\vec{b}}{|\vec{b}|}\times\vec{a} = \dfrac{\vec{b}\times\vec{a}}{|\vec{b}|} = \dfrac{19\vec{k}}{4.12} = 4.6\vec{k}$

$a_{\perp b} = |\vec{e_b}\times\vec{a}| = \sqrt{4.6^2} = 4.6$



q) $$\vec{a}_{\perp b} = (\vec{e_b}\times\vec{a})\times\vec{e_b} = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 4.6 \\ 0.97 & -0.24 & 0 \end{vmatrix} \end{array} =$$

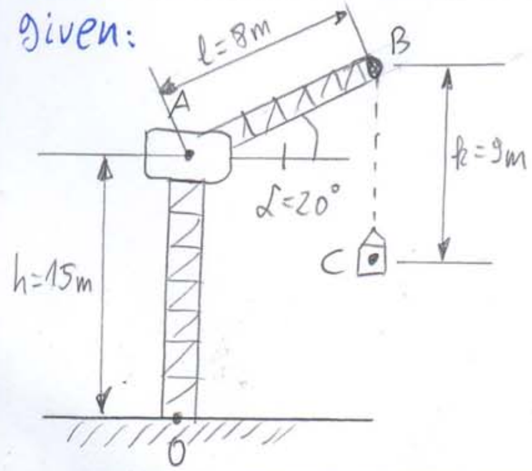
$= \vec{i}\,(0\cdot 0 - 4.6\cdot(-0.24)) - \vec{j}\,(0\cdot 0 - 4.6\cdot 0.97) + \vec{k}\cdot(0\cdot(-0.24) - 0\cdot 0.97) =$

$= (+1.11\vec{i} + 4.47\vec{j})$

Or: $\vec{a} = \vec{a}_{\parallel b} + \vec{a}_{\perp b} \Rightarrow \vec{a}_{\perp b} = \vec{a} - \vec{a}_{\parallel b} = (3\vec{i}+4\vec{j}) - (1.89\vec{i} - 0.47\vec{j}) = 1.11\vec{i} + 4.47\vec{j}$

EXERCISE 1.2

given:



task:

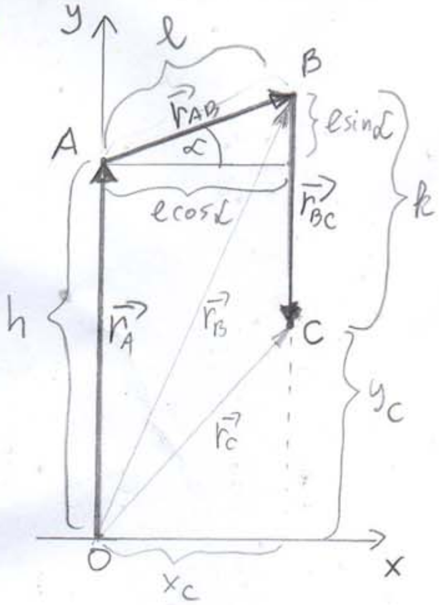
a) position vectors of point A, B and C

b) How high the load is? (point c)

c) How far is the load from the operator. In which direction does he see the load?

d) unit vector of AB

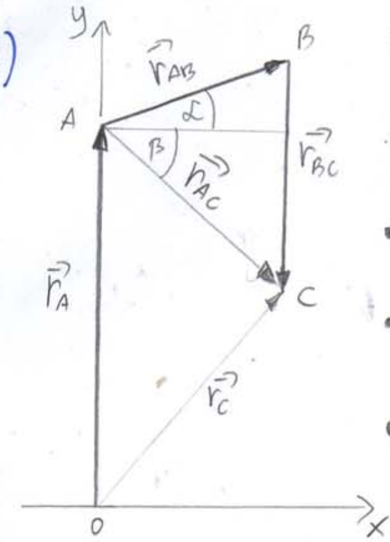
a)



- $\vec{r}_A = h\vec{j} = (15\vec{j})\,m$
  $\overset{\shortparallel}{(\vec{r}_{OA})}$
- $\vec{r}_{AB} = (l\cos\alpha)\vec{i} + (l\sin\alpha)\vec{j} =$
  $= (8\cos 20^\circ)\vec{i} + (8\sin 20^\circ)\vec{j} = (7.52\vec{i} + 2.74\vec{j})\,m$
- $\vec{r}_B = \vec{r}_A + \vec{r}_{AB} = 15\vec{j} + 7.52\vec{i} + 2.74\vec{j} =$
  $= (7.52\vec{i} + 17.74\vec{j})\,m$
- $\vec{r}_{BC} = -k\vec{j} = (-9\vec{j})\,m$
- $\vec{r}_C = \vec{r}_B + \vec{r}_{BC} = 7.52\vec{i} + 17.74\vec{j} + (-9\vec{j}) =$
  $= (\underbrace{7.52}_{x_C}\vec{i} + \underbrace{8.74}_{y_C}\vec{j})\,m$

b) $y_C = 8.74\,m$

c)



- $\vec{r}_{AC} = \vec{r}_C - \vec{r}_A = (7.52\vec{i} + 8.74\vec{j}) - 15\vec{j} = (7.52\vec{i} - 6.26\vec{j})\,m$
  or:
  $\vec{r}_{AC} = \vec{r}_{AB} + \vec{r}_{BC} = (7.52\vec{i} + 2.74\vec{j}) + (-9\vec{j}) = (7.52\vec{i} - 6.26\vec{j})\,m$
- $|\vec{r}_{AC}| = \sqrt{7.52^2 + 6.26^2} = 9.78\,m$
- $\vec{e}_{AC} = \dfrac{\vec{r}_{AC}}{|\vec{r}_{AC}|} = \dfrac{7.52\vec{i} - 6.26\vec{j}}{9.78} = 0.77\vec{i} - 0.64\vec{j}$
- $\beta = \operatorname{atan}\dfrac{6.26}{7.52} = \operatorname{atan}\dfrac{0.64}{0.77} = 39.78^\circ$

NO UNITS!

d) $\vec{e}_{AB} = \dfrac{\vec{r}_{AB}}{|\vec{r}_{AB}|} = \dfrac{\vec{r}_{AB}}{l} = \dfrac{7.52\vec{i} + 2.74\vec{j}}{8} = 0.94\vec{i} + 0.34\vec{j}$

or: $\vec{e}_{AB} = \cos\alpha\,\vec{i} + \sin\alpha\,\vec{j} = \cos 20^\circ\,\vec{i} + \sin 20^\circ\,\vec{j} = 0.94\vec{i} + 0.34\vec{j}$