FEM

CHAPTER III

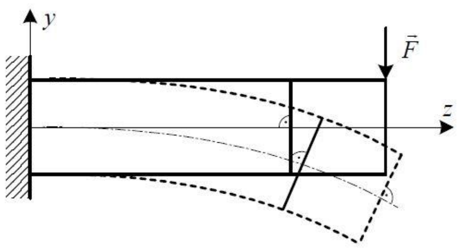
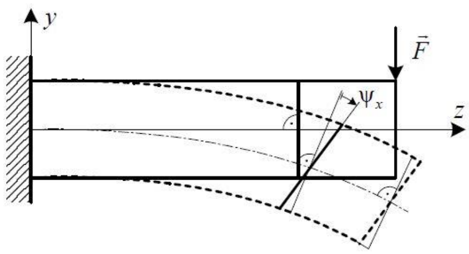
BEAMS

1. ABOUT BEAMS

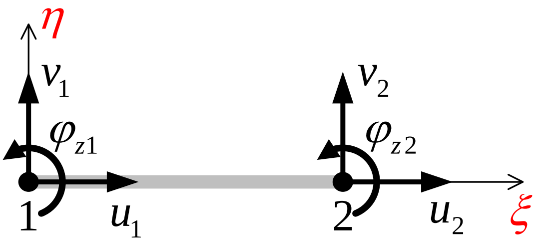
Beam elements are suited for situations wherein the rod may be subjected to loads that can create tension/compression, bending about different axes and torsion. Beam elements are typically used in frames, where the stuctural members are rigidly connected with welded joints or bolted joints. Consequently they play a significant role in many engineering applications including buildings, bridges, automobiles, and airplane structures. In case of a trusses all loads are assumed to apply at the joints of the truss structure. Therefore, no bending of the members are allowed. In case of beams, loads may be applied anywhere along the beam and the loading will create bending in the beam.



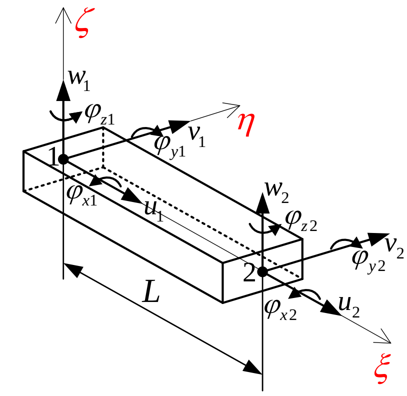
We distinguish 2 different beam theories:

Euler-Bernoulli beam theory	**Timoshenko beam theory**
It assumes that the cross-section of the beam is always perpendicular to the center line (also after the deformation). Deflection is calculated only using bending moment, without taking shear forces into account. It can be used to determine shear stresses but doesn't account for their influence on deflection.	It is the extended version of Euler-Bernoulli theory that takes into consideration deformations caused by shear. It assumes that the cross-section after deformation doesn't have to be perpendicular to the center line.
	
It is suitable for thin beams.	It is suitable for thick beams.

In 2D, each node has 3 degrees of freedom: displacements in ξ ,η directions and rotations about $\zeta(=z)$-axis. Therefore a two node beam element has 6 degrees of freedom, thus the elemental matrix is a 6X6 matrix.



In 3D, each node has 6 degrees of freedom: displacements in ξ ,η and ζ directions and rotations about ξ ,η and ζ -axes. Therefore a two node beam element has 12 degrees of freedom, thus the elemental matrix is a 12X12 matrix.



In the followings, the formulation of a 2D Euler-Bernoulli Beam element is presented.

2. 2D EULER-BERNOULLI BEAM FORMULATION

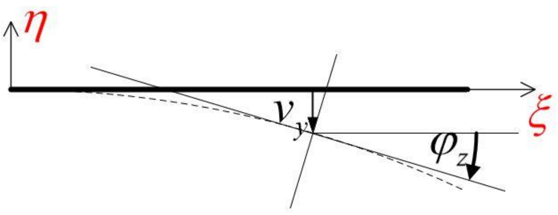
Independent fields:

$u(\xi)$

$v(\xi)$

According to Euler-Bernoully theory, the rotation about axis ζ is not independent, as it is the derivative of the vertical displacement function.

$$\varphi_z(\xi) = \frac{dv(\xi)}{d\xi}$$



Aprroximation of displacements and angular displacement:

$$u(\xi) = a_1 + a_2 x$$

$$v(\xi) = a_3 + a_4 x + a_5 x^2 + a_6 x^3$$

$$\varphi_z(\xi) = \frac{dv(\xi)}{d\xi} = a_4 + 2a_5 x + 3a_6 x^2$$

Utilizing nodal parameters:

$\xi = 0$	$\xi = L$
$u(\xi = 0) = a_1 + a_2 \cdot 0 = u_1$	$u(\xi = L) = a_1 + a_2 \cdot L = u_2$
$v(\xi = 0) = a_3 + a_4 \cdot 0 + a_5 \cdot 0^2 + a_6 \cdot 0^3 = v_1$	$v(\xi = L) = a_3 + a_4 \cdot L + a_5 \cdot L^2 + a_6 \cdot L^3 = v_2$
$\varphi_z(\xi = 0) = \frac{dv(\xi)}{\xi} = a_4 + 2a_5 \cdot 0 + 3a_6 \cdot 0^2 = \varphi_{z1}$	$\varphi_z(\xi = L) = \frac{dv(\xi)}{\xi} = a_4 + 2a_5 \cdot L + 3a_6 \cdot L^2 = \varphi_{z2}$

The above equation system in matrix form.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & L & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & L^2 & L^3 \\ 0 & 0 & 0 & 1 & 2L & 3L^2 \end{bmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}}_{\underline{a}} = \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ \varphi_{z1} \\ u_2 \\ v_2 \\ \varphi_{z2} \end{bmatrix}}_{\underline{u}}$$

$$\underline{\underline{A}}\,\underline{a} = \underline{u} \quad \Rightarrow \quad \underline{a} = \underline{\underline{A}}^{-1}\underline{u}$$

Solution of the equation system:

$$a_1 = u_1$$

$$a_2 = -\frac{u_1 - u_2}{L}$$

$$a_3 = v_1$$

$$a_4 = \varphi_{z1}$$

$$a_5 = -\frac{3u_2 - 3v_2 + 2L\varphi_{z1} + L\varphi_{z2}}{L^2}$$

$$a_6 = \frac{2v_1 - 2v_2 + L\varphi_{z1} + L\varphi_{z2}}{L^3}$$

Substituting the above expressions into the approximation functions:

$$u(\xi) = u_1 - \frac{u_1 - u_2}{L}\xi$$

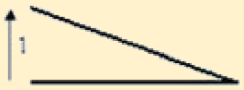
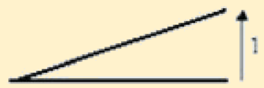
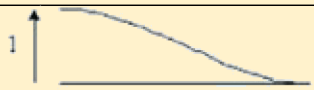
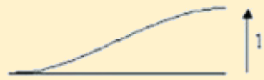
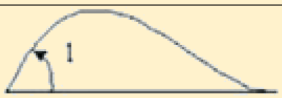
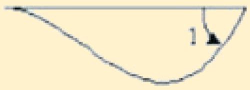
$$u(\xi) = u_1 - \frac{u_1}{L}\xi + \frac{u_2}{L}\xi$$

$$u(\xi) = \underbrace{\left(1 - \frac{1}{L}\xi\right)}_{N_1(\xi)} u_1 + \underbrace{\left(\frac{1}{L}\xi\right)}_{N_2(\xi)} u_2$$

$$v(\xi) = v_1 + \varphi_{z1}\xi - \frac{3v_1 - 3v_2 + 2L\varphi_{z1} + L\varphi_{z2}}{L^2}\xi^2 + \frac{2v_1 - 2v_2 + L\varphi_{z1} + L\varphi_{z2}}{L^3}\xi^3$$

$$v(\xi) = u_2 + \varphi_{z1}\xi - \frac{3v_1}{L^2}\xi^2 + \frac{3v_2}{L^2}\xi^2 - \frac{2L\varphi_{z1}}{L^2}\xi^2 - \frac{L\varphi_{z2}}{L^2}\xi^2 + \frac{2v_1}{L^3}\xi^3 - \frac{2v_2}{L^3}\xi^3 + \frac{L\varphi_{z1}}{L^3}\xi^3 + \frac{L\varphi_{z2}}{L^3}\xi^3$$

$$v(\xi) = \underbrace{\left(1 - \frac{3}{L^2}\xi^2 + \frac{2}{L^3}\xi^3\right)}_{N_3(\xi)} v_1 + \underbrace{\left(\xi - \frac{2}{L}\xi^2 + \frac{1}{L^2}\xi^3\right)}_{N_4(\xi)} \varphi_{z1} + \underbrace{\left(\frac{3}{L^2}\xi^2 - \frac{2}{L^3}\xi^3\right)}_{N_5(\xi)} v_2 + \underbrace{\left(-\frac{1}{L}\xi^2 + \frac{1}{L^2}\xi^3\right)}_{N_6(\xi)} \varphi_{z2}$$

SHAPE FUNCTIONS OF A 2D BEAM ELEMENT	
 $N_1(\xi) = 1 - \frac{1}{L}\xi$	 $N_2(\xi) = \frac{1}{L}\xi$
 $N_3(\xi) = 1 - \frac{3}{L^2}\xi^2 + \frac{2}{L^3}\xi^3$	 $N_5(\xi) = \frac{3}{L^2}\xi^2 - \frac{2}{L^3}\xi^3$
 $N_4(\xi) = \xi - \frac{2}{L}\xi^2 + \frac{1}{L^2}\xi^3$	 $N_6(\xi) = -\frac{1}{L}\xi^2 + \frac{1}{L^2}\xi^3$

The displacements with shape function matrix and nodal displacement vector:

$$\underbrace{\begin{bmatrix} u(\xi) \\ v(\xi) \end{bmatrix}}_{\underline{u}(\xi)} = \underbrace{\begin{bmatrix} N_1(\xi) & 0 & 0 & N_2(\xi) & 0 & 0 \\ 0 & N_3(\xi) & N_4(\xi) & 0 & N_5(\xi) & N_6(\xi) \end{bmatrix}}_{\underline{\underline{N}}(\xi)} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ \varphi_{z1} \\ u_2 \\ v_2 \\ \varphi_{z2} \end{bmatrix}}_{\underline{q}}$$

Strain vector:

$$\begin{bmatrix} \varepsilon(\xi) \\ \kappa(\xi) \end{bmatrix} = \begin{bmatrix} \frac{du(\xi)}{d\xi} \\ \frac{d^2 v(\xi)}{d\xi^2} \end{bmatrix} = \begin{bmatrix} \frac{d}{d\xi} & 0 \\ 0 & \frac{d^2}{d\xi^2} \end{bmatrix} \begin{bmatrix} u(\xi) \\ v(\xi) \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon(\xi) \\ \kappa(\xi) \end{bmatrix} = \begin{bmatrix} \frac{d}{d\xi} & 0 \\ 0 & \frac{d^2}{d\xi^2} \end{bmatrix} \begin{bmatrix} N_1(\xi) & 0 & 0 & N_2(\xi) & 0 & 0 \\ 0 & N_3(\xi) & N_4(\xi) & 0 & N_5(\xi) & N_6(\xi) \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \varphi_{z1} \\ u_2 \\ v_2 \\ \varphi_{z2} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \varepsilon(\xi) \\ \kappa(\xi) \end{bmatrix}}_{\underline{\underline{\varepsilon}}(\xi)} = \underbrace{\begin{bmatrix} \frac{dN_1(\xi)}{d\xi} & 0 & 0 & \frac{dN_2(\xi)}{d\xi} & 0 & 0 \\ 0 & \frac{d^2 N_3(\xi)}{d\xi^2} & \frac{d^2 N_4(\xi)}{d\xi^2} & 0 & \frac{d^2 N_5(\xi)}{d\xi^2} & \frac{d^2 N_6(\xi)}{d\xi^2} \end{bmatrix}}_{\underline{\underline{B}}(\xi)} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ \varphi_{z1} \\ u_2 \\ v_2 \\ \varphi_{z2} \end{bmatrix}}_{\underline{\underline{q}}}$$

Inrenal forces/moments:

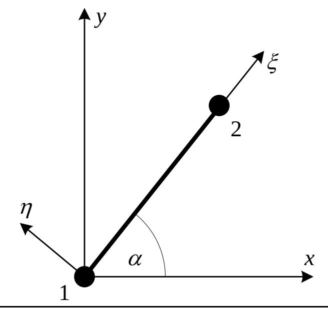
$$\begin{bmatrix} N(\xi) \\ M_{bz}(\xi) \end{bmatrix} = \begin{bmatrix} AE\varepsilon(\xi) \\ I_z E\kappa(\xi) \end{bmatrix} = \underbrace{\begin{bmatrix} AE & 0 \\ 0 & I_z E \end{bmatrix}}_{\underline{\underline{D}}} \begin{bmatrix} \varepsilon(\xi) \\ \kappa(\xi) \end{bmatrix}$$

The stiffness matrix in local coordinate system:	The nodal load vector from line loads:
$\underline{\underline{K}}_0 = \int_0^L \underline{\underline{B}}^T \underline{\underline{D}}\underline{\underline{B}} d\xi$ $\underline{\underline{K}}_0 = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12I_zE}{L^3} & \frac{6I_zE}{L^2} & 0 & -\frac{12I_zE}{L^3} & \frac{6I_zE}{L^2} \\ 0 & \frac{6I_zE}{L^2} & \frac{4I_zE}{L} & 0 & -\frac{6I_zE}{L^2} & \frac{2I_zE}{L} \\ -\frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12I_zE}{L^3} & -\frac{6I_zE}{L^2} & 0 & \frac{12I_zE}{L^3} & -\frac{6I_zE}{L^2} \\ 0 & \frac{6I_zE}{L^2} & \frac{2I_zE}{L} & 0 & -\frac{6I_zE}{L^2} & \frac{4I_zE}{L} \end{bmatrix}$	$\underline{\underline{f}} = \int_0^L \underline{\underline{N}}^T d\xi \begin{bmatrix} f_\xi \\ f_\eta \end{bmatrix}$ $\underline{\underline{f}} = \begin{bmatrix} \frac{Lf_\xi}{2} \\ \frac{Lf_\eta}{2} \\ \frac{L^2 f_\eta}{12} \\ \frac{Lf_\xi}{2} \\ \frac{Lf_\eta}{2} \\ -\frac{L^2 f_\eta}{12} \end{bmatrix}$

Matlab code for obtaining stiffness matrix and nodal load vector paramtetrically

```
clear all
clc
% Parameter set %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
nset = [2,4];
% initialization %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
n = sum(nset);
a=sym('a', [n,1]);
v_aX=sym('v_aX', [3,1]);
v_X=sym('v_X',[2,1]);
v_a=sym('v_a',[n,1]);
A0=sym('A0', [n,n]);
Amat=sym('A',[2,n]);
B=sym('B',[2,n]);
u=sym('u', [n,1]);
syms X L A Iz E qX qY;
% Approximation functions %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
v_aX(1)    = a(1)+a(2)*X;
v_aX(2)    = a(3)+a(4)*X+a(5)*X^2+a(6)*X^3;
% Angular displacements:
v_aX(3) =diff(v_aX(2),X);
% Nodal parameters:
for i=1:3
    v_a(i)=subs(v_aX(i),X,0);
    v_a(3+i)=subs(v_aX(i),X,L);
end
% Coefficient matrix:
for i=1:n
  for j=1:n
    A0(i,j)=diff(v_a(i),a(j));
  end
end
% Solving the equation -> a parameters with v nodal displacements
a0=A0\u;
% Substituting a parameters with u nodal displacements into
original approximations
v_X(1) = subs(v_aX(1),a(1:nset(1)),a0(1:nset(1)));
v_X(2) =
subs(v_aX(2),a(nset(1)+1:nset(1)+nset(2)),a0(nset(1)+1:nset(1)+nset
(2)));
% Shape Function matrix:
for i=1:2
    for j=1:n
        Amat(i,j)=diff(v_X(i),u(j));
    end
end
% B matrix:
B(1,:)=diff(Amat(1,:),X);
B(2,:)=diff(Amat(2,:),X,2);
% Property matrix:
D = diag([A*E;Iz*E]);
% Stiffness matrix and nodal load vector:
K1=int(B'*D*B,[0,L])
f=int(Amat',[0,L])*[qX;qY]
```

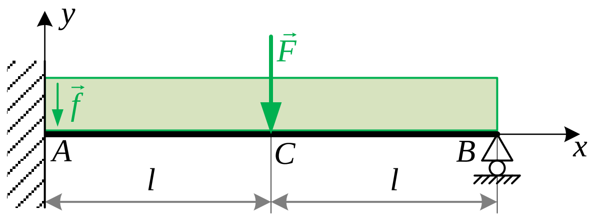
Transformation matrix:

	$\underline{\underline{T}} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & \cos\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

The stiffness matrix in global xy coordinate system:

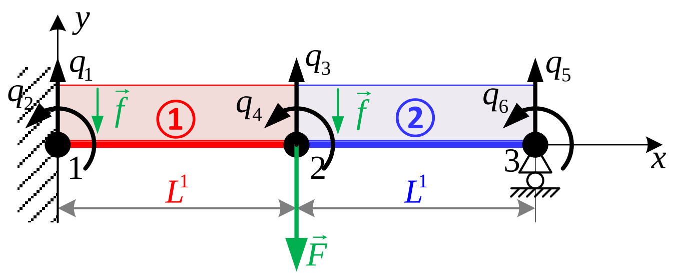
$$\underline{\underline{K}} = \underline{\underline{T}}^T \underline{\underline{K}}_0 \underline{\underline{T}}$$

3. NUMERICAL EXAMPLE

GIVEN:		TASK:
	$I_z = 6\cdot 10^4\ \mathrm{mm}^4$ $E = 2\cdot 10^5\ \mathrm{MPa}$ $\vec{F} = \left(-198\,\vec{j}\right)\mathrm{N}$ $\vec{f} = \left(-0.09\,\vec{j}\right)\dfrac{\mathrm{N}}{\mathrm{mm}}$ $l = 1000\ \mathrm{mm}$	vertical displacements and angular displacements of cross sections A, B and C Use 2 beam elements!

FEM model:

- Node numbers: **1, 2, 3**
- Element numbers: ①, ②
- Length of the beam elements $L^1 = L^2 = l = 1000\ mm$
- Coordinate systems: $\xi^1 = \xi^2 = x$, $\eta^1 = \eta^2 = y$ → The local and global coordinate systems are the same → no coordinate transformation is required.
- There is no load in the x direction, therefore $u(x) = 0$. The points on the centre line of the beam do not move in x direction, no normal force occurs in the structure. Because of this, we have 2 DOF at each node (vertical displacement and angular displacement) and as a result from this the element stiffness matrix is a 4x4 matrix.



nodal displacement vector:

$$\underline{\underline{q}} = \begin{bmatrix} v_1 \\ \varphi_1 \\ v_2 \\ \varphi_2 \\ v_3 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}$$

- DOF connectivity: $\underline{\underline{Dcon}} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

- Element stiffness matrix: $\underline{\underline{K}}^1 = \underline{\underline{K}}^2 = \dfrac{I_z E}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$

- Global stiffness matrix:

$$\underline{\underline{K}} = \underline{\underline{K}}^1 + \underline{\underline{K}}^2 = \frac{I_z E}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12 & -6L & 0 & 0 \\ 6L & 2L^2 & -6L & 4L^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{I_z E}{L^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 6L & -12 & 6L \\ 0 & 0 & 6L & 4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$\underline{\underline{K}} = \underline{\underline{K}}^1 + \underline{\underline{K}}^2 = \frac{I_z E}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$\underline{\underline{K}} = \frac{I_z E}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Element nodal load vector:	Global nodal load vector from line load:
$\underline{\underline{f}}_q^1 = \underline{\underline{f}}_q^2 = \begin{bmatrix} \frac{Lf}{2} \\ \frac{L^2 f}{12} \\ \frac{Lf}{2} \\ -\frac{L^2 f}{12} \end{bmatrix}$	$\underline{\underline{f}}_q = \underline{\underline{f}}_q^1 + \underline{\underline{f}}_q^2 = \begin{bmatrix} \frac{Lf}{2} \\ \frac{L^2 f}{12} \\ \frac{Lf}{2} \\ -\frac{L^2 f}{12} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{Lf}{2} \\ \frac{L^2 f}{12} \\ \frac{Lf}{2} \\ -\frac{L^2 f}{12} \end{bmatrix} = \begin{bmatrix} \frac{Lf}{2} \\ \frac{L^2 f}{12} \\ \frac{Lf}{2}+\frac{Lf}{2} \\ -\frac{L^2 f}{12}+\frac{L^2 f}{12} \\ \frac{Lf}{2} \\ -\frac{L^2 f}{12} \end{bmatrix} = \begin{bmatrix} \frac{Lf}{2} \\ \frac{L^2 f}{12} \\ Lf \\ 0 \\ \frac{Lf}{2} \\ -\frac{L^2 f}{12} \end{bmatrix}$

Global nodal load vevtor from concentrated forces/moments:

$$\underline{\underline{f}}_{FM} = \begin{bmatrix} 0 \\ 0 \\ F \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Global nodal load vector:

$$\underline{\underline{f}} = \underline{\underline{f}}_q + \underline{\underline{f}}_{FM} = \begin{bmatrix} \frac{Lf}{2} \\ \frac{L^2 f}{12} \\ Lf \\ 0 \\ \frac{Lf}{2} \\ -\frac{L^2 f}{12} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ F \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{Lf}{2} \\ \frac{L^2 f}{12} \\ Lf + F \\ 0 \\ \frac{Lf}{2} \\ -\frac{L^2 f}{12} \end{bmatrix}$$

The basic $\underline{\underline{K}}\underline{\underline{q}} = \underline{\underline{f}}$ equation:

$$\frac{I_z E}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} Lf/2 \\ L^2 f/12 \\ Lf + F \\ 0 \\ Lf/2 \\ -L^2 f/12 \end{bmatrix}$$

Taking into account the constraints:

- Because of the fixed support at the left end, the both the vertical displacement and the angular displacement of the first node is zero: $q_1 = 0$, $q_2 = 0$
- Because of the roller support at the right end, the vertical displacement of the third node is zero: $q_5 = 0$

Thus the 1st, 2nd and 5th rows and columns can be deleted:

$$\frac{I_z E}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} Lf/2 \\ L^2 f/12 \\ Lf+F \\ 0 \\ Lf/2 \\ -L^2 f/12 \end{bmatrix}$$

The system of equations - with the condensed matrices - to solve:

$$\frac{I_z E}{L^3}\begin{bmatrix} 24 & 0 & 6L \\ 0 & 8L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix}\begin{bmatrix} q_3 \\ q_4 \\ q_6 \end{bmatrix} = \begin{bmatrix} Lf+F \\ 0 \\ -L^2 f/12 \end{bmatrix}$$

After substitution:

$$\frac{6\cdot 10^4 \cdot 2\cdot 10^5}{1000^3}\begin{bmatrix} 24 & 0 & 6\cdot 1000 \\ 0 & 8\cdot 1000^2 & 2\cdot 1000^2 \\ 6\cdot 1000 & 2\cdot 1000^2 & 4\cdot 1000^2 \end{bmatrix}\begin{bmatrix} q_3 \\ q_4 \\ q_6 \end{bmatrix} = \begin{bmatrix} 1000\cdot(-0.09)+(-198) \\ 0 \\ -1000^2\cdot(-0.09)/12 \end{bmatrix}$$

$$\begin{bmatrix} 288 & 0 & 72\,000 \\ 0 & 96\,000\,000 & 24\,000\,000 \\ 72\,000 & 24\,000\,000 & 48\,000\,000 \end{bmatrix}\begin{bmatrix} q_3 \\ q_4 \\ q_6 \end{bmatrix} = \begin{bmatrix} -288 \\ 0 \\ 7\,500 \end{bmatrix}$$

Rewritten the above matrix equation in a scalar systems of equations form:

$$\left.\begin{aligned} &1)\quad 288q_3 + 72\,000q_6 = -288 \\ &2)\quad 96\,000\,000q_4 + 24\,000\,000q_6 = 0 \\ &3)\quad 72\,000q_3 + 24\,000\,000q_4 + 48\,000\,000q_6 = 7\,500 \end{aligned}\right\}$$

$$1) \Rightarrow q_3 = \frac{-288 - 72\,000q_6}{288} = -1 - 250q_6$$

$$2) \Rightarrow q_4 = \frac{-24\,000\,000q_6}{96\,000\,000} = -0.25q_6$$

$$\begin{aligned} 3)\quad 72\,000\cdot(-1-250q_6) + 24\,000\,000\cdot(-0.25q_6) + 48\,000\,000q_6 &= 7\,500 \\ -72\,000 - 18\,000\,000q_6 - 6\,000\,000q_6 + 48\,000\,000q_6 &= 7\,500 \\ 24\,000\,000q_6 &= 79\,500 \\ q_6 &= 0.0033125 \end{aligned}$$

$$1) \Rightarrow q_3 = -1 - 250q_6 = -1 - 250\cdot 0.0033125 = -1.828125 \text{ mm}$$

$$2) \Rightarrow q_4 = -0.25q_6 = -0.25\cdot 0.0033125 = -0.000828125$$

Results:

$$q_3 = v_2 = -1.828125\,\mathrm{mm}$$

$$q_4 = \varphi_2 = -0.000828125\,(rad) \Rightarrow \varphi_2 = -0.000828125 \cdot \frac{180}{\pi} = -0.047448^{\circ}$$

$$q_6 = \varphi_3 = 0.0033125\,(rad) \Rightarrow \varphi_3 = 0.0033125 \cdot \frac{180}{\pi} = 0.18979^{\circ}$$

	cross section A	**cross section B**	**cross section C**
vertical displacement [mm]	0	-1.828125	0
angular displacement [deg]	0	-0.047448	0.18979

Visual representation of the displacement: