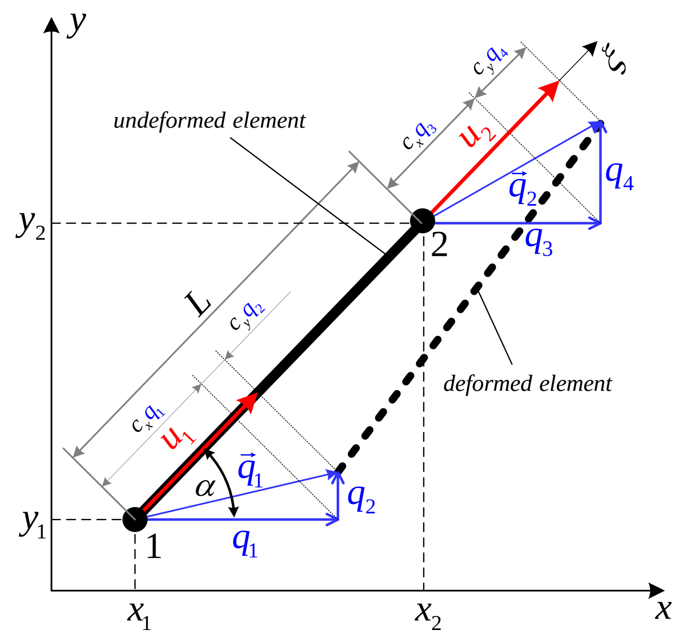
2D & 3D TRUSS ELEMENT

STIFFNESS MATRIX OF A 2D & 3D TRUSS ELEMENT



local coordinates (local DOF):

$$\underline{\underline{u}} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

global coordinates (global DOF):

$$\underline{\underline{q}} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

***Obtaining the transformation matrix:**

The local coordinates are the projections of the global displacements vectors onto axis ξ so they can be obtained with dot products:

$$u_1 = \vec{e} \cdot \vec{q}_1 = \begin{bmatrix} c_x & c_y \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$u_2 = \vec{e} \cdot \vec{q}_2 = \begin{bmatrix} c_x & c_y \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix}$$

in one equation:

STEPS:

1) direction vector of the bar element:

$$\vec{r}_{12} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}$$

2) length of the bar element:

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

3) unit direction vector of the bar element:

$$\vec{e} = \frac{\vec{r}_{12}}{L} = c_x \vec{i} + c_y \vec{j}$$

4) 2D transformation matrix *:

$$\underline{\underline{T}} = \begin{bmatrix} c_x & c_y & 0 & 0 \\ 0 & 0 & c_x & c_y \end{bmatrix}$$

5) stiffness matrix in 1D:

$$\underline{\underline{K}}_0 = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

6) transformed 2D stiffness matrix:

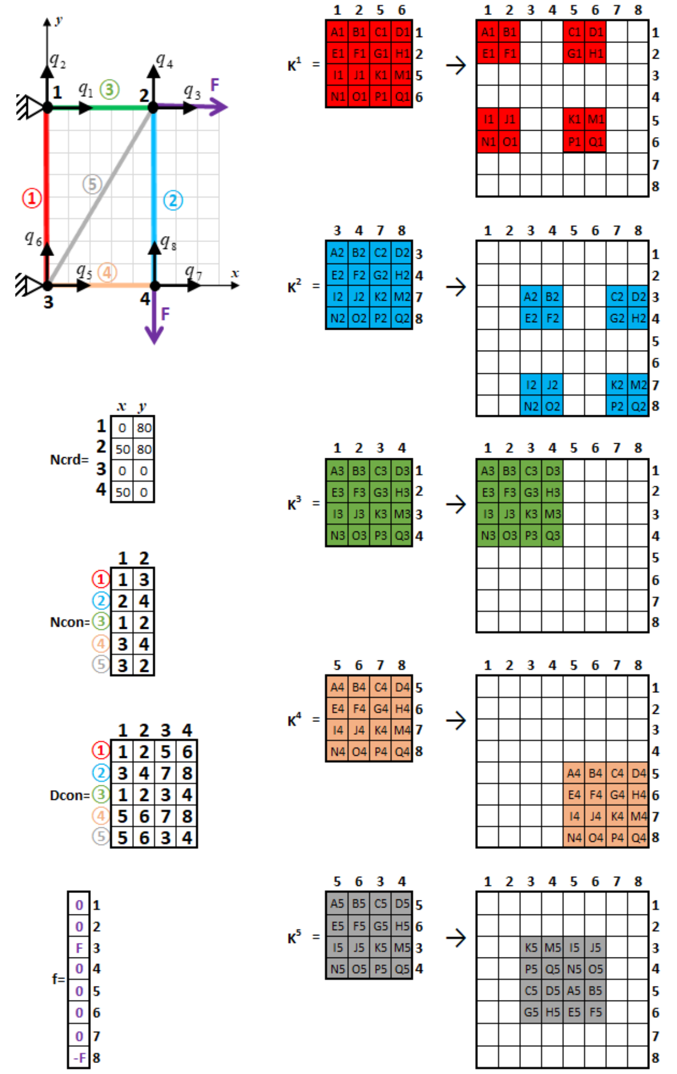
$$\underline{\underline{K}} = \underline{\underline{T}}^T \underline{\underline{K}}_0 \underline{\underline{T}} =$$

$$= \begin{bmatrix} c_x & 0 \\ c_y & 0 \\ 0 & c_x \\ 0 & c_y \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_x & c_y & 0 & 0 \\ 0 & 0 & c_x & c_y \end{bmatrix} =$$

$$\underline{\underline{K}} = \frac{AE}{L} \begin{bmatrix} c_x^2 & c_x c_y & -c_x^2 & -c_x c_y \\ c_x c_y & c_y^2 & -c_x c_y & -c_y^2 \\ -c_x^2 & -c_x c_y & c_x^2 & c_x c_y \\ -c_x c_y & -c_y^2 & c_x c_y & c_y^2 \end{bmatrix}$$

$\underset{\underline{\underline{u}}}{\begin{bmatrix}u_1\\u_2\end{bmatrix}}=\underbrace{\begin{bmatrix}c_x & c_y & 0 & 0\\0 & 0 & c_x & c_y\end{bmatrix}}_{\underline{\underline{T}}}\underset{\underline{\underline{q}}}{\begin{bmatrix}q_1\\q_2\\q_3\\q_4\end{bmatrix}}$	
In a 3D case:	
$\underline{\underline{u}}=\begin{bmatrix}u_1\\u_2\end{bmatrix} \qquad \underline{\underline{q}}=\begin{bmatrix}q_1\\q_2\\q_3\\q_4\\q_5\\q_6\end{bmatrix}$	$\vec{r}_{12}=(x_2-x_1)\vec{i}+(y_2-y_1)\vec{j}+(z_2-z_1)\vec{k}$ $L=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$ $\vec{e}=\frac{\vec{r}_{12}}{L}=c_x\vec{i}+c_y\vec{j}+c_z\vec{k}$ $\underline{\underline{T}}=\begin{bmatrix}c_x & c_y & c_z & 0 & 0 & 0\\0 & 0 & 0 & c_x & c_y & c_z\end{bmatrix}$ $\underline{\underline{K}}=\underline{\underline{T}}^T\underline{\underline{K}}_0\underline{\underline{T}}=$

ASSEMBLING GLOBAL STIFFNESS MATRIX AND NODAL LOAD VECTOR



Ncrd=

	x	y
1	0	80
2	50	80
3	0	0
4	50	0

Ncon=

	1	2
①	1	3
②	2	4
③	1	2
④	3	4
⑤	3	2

Dcon=

	1	2	3	4
①	1	2	5	6
②	3	4	7	8
③	1	2	3	4
④	5	6	7	8
⑤	5	6	3	4

f=

0	1
0	2
F	3
0	4
0	5
0	6
0	7
-F	8

$$K^1 = \begin{array}{cccc|c} 1 & 2 & 5 & 6 & \\ A1 & B1 & C1 & D1 & 1 \\ E1 & F1 & G1 & H1 & 2 \\ I1 & J1 & K1 & M1 & 5 \\ N1 & O1 & P1 & Q1 & 6 \end{array}$$

$$K^2 = \begin{array}{cccc|c} 3 & 4 & 7 & 8 & \\ A2 & B2 & C2 & D2 & 3 \\ E2 & F2 & G2 & H2 & 4 \\ I2 & J2 & K2 & M2 & 7 \\ N2 & O2 & P2 & Q2 & 8 \end{array}$$

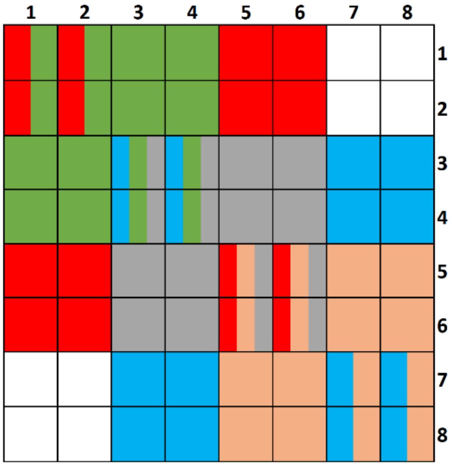
$$K^3 = \begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ A3 & B3 & C3 & D3 & 1 \\ E3 & F3 & G3 & H3 & 2 \\ I3 & J3 & K3 & M3 & 3 \\ N3 & O3 & P3 & Q3 & 4 \end{array}$$

$$K^4 = \begin{array}{cccc|c} 5 & 6 & 7 & 8 & \\ A4 & B4 & C4 & D4 & 5 \\ E4 & F4 & G4 & H4 & 6 \\ I4 & J4 & K4 & M4 & 7 \\ N4 & O4 & P4 & Q4 & 8 \end{array}$$

$$K^5 = \begin{array}{cccc|c} 5 & 6 & 3 & 4 & \\ A5 & B5 & C5 & D5 & 5 \\ E5 & F5 & G5 & H5 & 6 \\ I5 & J5 & K5 & M5 & 3 \\ N5 & O5 & P5 & Q5 & 4 \end{array}$$

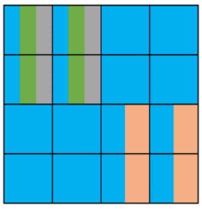
Pattern of the global stiffness matrix:

$\underline{\underline{K}} =$



After applying **boundary conditions**, the 1st, 2nd, 5th and 6th rows and columns can be deleted. The pattern of the condensed stiffness matrix and condensed nodal load vector:

$\underline{\underline{K}} =$



$\underline{f} =$

F
0
0
-F

2.3. CALCULATING NODAL DISPLACEMENTS

$$\underline{\underline{K}}\,\underline{q} = \underline{f} \quad \Rightarrow \quad \underline{q} = \underline{\underline{K}}^{-1}\,\underline{f}$$

2.4 CALCULATING OTHER QUANTITES FROM DISPLACEMENTS

- Nodal forces (loads and reaction forces)

$$\underline{\underline{f}} = \underline{\underline{K}}\,\underline{\underline{q}}$$

- Change of length of elements:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} c_x & c_y & 0 & 0 \\ 0 & 0 & c_x & c_y \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

$$\Delta L = u_2 - u_1$$

(1 and 2 are local node numbers)

- Normal strains of elements:

$$\varepsilon_\xi = \frac{\Delta L}{L}$$

- Normal stress in elements:

$$\sigma_\xi = E\varepsilon_\xi$$

- Normal forces in elements:

$$N = \sigma_\xi A$$

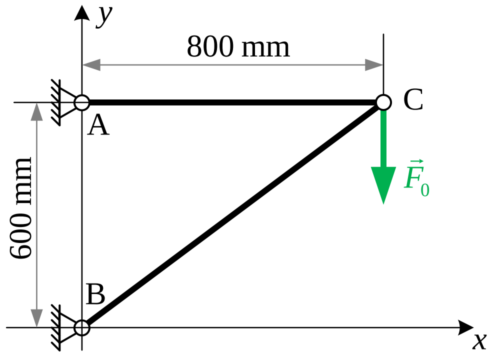
MATLAB CODE FOR TRUSS PROBLEMS

MATLAB/OCTAVE code for solving any 2D / 3D truss problems: (presented through the above example)

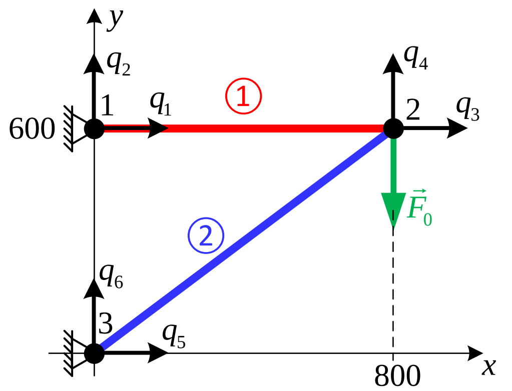
```
clear all           % clearing memory
clc                 % clearing Command Window
format shortG       % Command window display format
% pkg load signal % (only needed for Octave)

%% PREPROCESSING
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% properties (property ID, cross section area, Young's modulus):
prp=[1 100 200000];

% nodal coordinates (x y (z)):
crd=[0   800
     500 800
     0   0
     500 0];

% connectivity matrix with poperty ID-s
con=[1 3 1
     2 4 1
     1 2 1
     3 4 1
     3 2 1];

% constraints (kinematic boundary condition):
kbc=[1 1
     1 2
     3 1
     3 2];

% loads (dynamic boundary condition):
dbc=[2 1 20000
     4 2 -20000];

% display settings:
sc=200;     % enlargement of displacements

%% SOLVING
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
dim=size(crd,2);            % dimension (2D or 3D)
non=size(crd,1);            % number of nodes
noe=size(con,1);            % number of elements
dof=non*dim;                % degrees of freedom of the structure

K=zeros(dof,dof);           % initialization of globall dtiffness matrix
F=zeros(dof,1);             % initialization of global nodal load vector
F((dbc(:,1)-1).*dim+dbc(:,2))=dbc(:,3); % inserting known loads
q=zeros(dof,1);             % initialization of nodal displacement vector

Pset=(kbc(:,1)-1).*dim+kbc(:,2); % constrained dipslacement coordinates
Nset=setdiff([1:dof]',Pset);     % free displacement coordinates

N1=con(:,1);          % node numbers at the beginning of the elements
N2=con(:,2);                % node numbers at the end of the elements
A=prp(con(:,3),2);          % cross section areas of the elements
E=prp(con(:,3),3);          % Young's modulus of the elements
```

```
v=crd(N2,:)-crd(N1,:) ;     % direction vectors of elements
L=rssq(v,2);                % lenghts of elemets
uv=v./L;                    % direction unit vectors of elements

% assembling stiffness matrix:
for e=1:noe
    T=blkdiag(uv(e,:),uv(e,:));     % transformation matrix
    K1=A(e)*E(e)/L(e);              % stiffness matrix elements
    Ke=T'*[K1 -K1;-K1 K1]*T;        % transformed stiffness matrix
    d1=(N1(e)-1)*dim+1:N1(e)*dim;   % DOF at the beginning
    d2=(N2(e)-1)*dim+1:N2(e)*dim;   % DOF at the end
    d=[d1,d2];                      % DOF of the eth element
    K(d,d)=K(d,d)+Ke;               % inserting to global stiffness
matrix
end

% applying kinematic boundary conditions:
Knn=K(Nset,Nset);                   % condensed stiffness matrix
Fn=F(Nset);                         % condensed nodal load vector

% calculation of nodal displacements
q(Nset)=Knn\Fn;                     % solving Kq=F equation

%% POSTPROCESSING
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% calculating other quantities from displacements:
Fall=round(K*q.*1e8)./1e8;      % nodal forces (loads and reaction
forces)
qmat=reshape(q,dim,non)'               % nodal displacement matrix
qabs=rssq(qmat,2)                      % displacement magnitudes
Fmat=reshape(Fall,dim,non)'            % nodal force matrix
DL=sum(uv.*(qmat(N2,:)-qmat(N1,:)),2); % elongation of elements
DL=round(DL.*1e8)./1e8;                % roundign elongations
EPS=DL./L;                             % normal strains of elements
S=E.*EPS                               % normal stresses in the elements
N=A.*S                                 % normal forces in the elements

% coordinates after deformation for plotting:
dcrd=crd+sc*qmat;
def(1:3:3*noe-2,:)=dcrd(N1,:);
def(2:3:3*noe-1,:)=dcrd(N2,:);
def(3:3:3*noe,:)=NaN;

% plotting:
if dim==2 % 2D case
    plot(def(:,1),def(:,2),'LineWidth',3);
else % 3D case
    plot3(def(:,1),def(:,2),def(:,3),'LineWidth',3);
end
axis equal
```

NUMERICAL EXAMPLE

GIVEN:		TASK:
	$A = 100\ \text{mm}^2$ $E = 200\,000\ \text{MPa}$ $\vec{F}_0 = (-18\,000\,\vec{j})\ N$	• displacement of point C • change of length of rods • normal strains of rods • normal stresses arising in the rods • normal forces in the rods

Finite element model:



Coordinate matrix:

$$\underline{\underline{Ncrd}} = \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 0 & 600 \\ 800 & 600 \\ 0 & 0 \end{bmatrix} \end{array} \text{mm}$$

Nodal connecctivity:

$$\underline{\underline{Ncon}} = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \end{array}$$

DOF connectivity:

$$\underline{\underline{Dcon}} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 3 & 4 \end{bmatrix} \end{array}$$

ELEMENT ①	**ELEMENT ②**
unit direction vector: $\vec{e} = \underbrace{1}_{c_x}\vec{i} + \underbrace{0}_{c_y}\vec{j}$	unit direction vector: $\vec{r}_{12} = (800\vec{i} + 600\vec{j})\ \text{mm}$ $L = \sqrt{800^2 + 600^2} = 1000\ \text{mm}$ $\vec{e} = \dfrac{\vec{r}_{12}}{L} = \dfrac{800\vec{i} + 600\vec{j}}{1000} = \underbrace{0.8}_{c_x}\vec{i} + \underbrace{0.6}_{c_z}\vec{j}$
transformation matrix: $\underline{\underline{T}} = \begin{bmatrix} c_x & c_y & 0 & 0 \\ 0 & 0 & c_x & c_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	transformation matrix: $\underline{\underline{T}} = \begin{bmatrix} c_x & c_y & 0 & 0 \\ 0 & 0 & c_x & c_y \end{bmatrix} = \begin{bmatrix} 0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.8 & 0.6 \end{bmatrix}$

stiffness matrix:

$$\underline{\underline{K}} = \begin{bmatrix} c_x & 0 \\ c_y & 0 \\ 0 & c_x \\ 0 & c_y \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_x & c_y & 0 & 0 \\ 0 & 0 & c_x & c_y \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} c_x^2 & c_x c_y & -c_x^2 & -c_x c_y \\ c_x c_y & c_y^2 & -c_x c_y & -c_y^2 \\ -c_x^2 & -c_x c_y & c_x^2 & c_x c_y \\ -c_x c_y & -c_y^2 & c_x c_y & c_y^2 \end{bmatrix}$$

stiffness matrix:

$$\frac{AE}{L}=\frac{100\cdot 200000}{800}=25000\frac{\mathrm{N}}{\mathrm{mm}}$$

$$\begin{bmatrix} c_x^2 & c_xc_y & -c_x^2 & -c_xc_y \\ c_xc_y & c_y^2 & -c_xc_y & -c_y^2 \\ -c_x^2 & -c_xc_y & c_x^2 & c_xc_y \\ -c_xc_y & -c_y^2 & c_xc_y & c_y^2 \end{bmatrix}=$$

$$=\begin{bmatrix} 1^2 & 1\cdot 0 & -1^2 & -1\cdot 0 \\ 1\cdot 0 & 0^2 & -1\cdot 0 & -0^2 \\ -1^2 & -1\cdot 0 & 01^2 & 1\cdot 0 \\ -1\cdot 0 & -0^2 & 1\cdot 0 & 0^2 \end{bmatrix}=$$

$$=\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{K}}=\begin{bmatrix} 25000 & 0 & -25000 & 0 \\ 0 & 0 & 0 & 0 \\ -25000 & 0 & 25000 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

stiffness matrix:

$$\frac{AE}{L}=\frac{100\cdot 200000}{1000}=20000\frac{\mathrm{N}}{\mathrm{mm}}$$

$$\begin{bmatrix} c_x^2 & c_xc_y & -c_x^2 & -c_xc_y \\ c_xc_y & c_y^2 & -c_xc_y & -c_y^2 \\ -c_x^2 & -c_xc_y & c_x^2 & c_xc_y \\ -c_xc_y & -c_y^2 & c_xc_y & c_y^2 \end{bmatrix}=$$

$$=\begin{bmatrix} 0.8^2 & 0.8\cdot 0.6 & -0.8^2 & -0.8\cdot 0.6 \\ 0.8\cdot 0.6 & 0.6^2 & -0.8\cdot 0.6 & -0.6^2 \\ -0.8^2 & -0.8\cdot 0.6 & 0.8^2 & 0.8\cdot 0.6 \\ -0.8\cdot 0.6 & -0.6^2 & 0.8\cdot 0.6 & 0.6^2 \end{bmatrix}=$$

$$=\begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix}$$

$$\underline{\underline{K}}=\begin{bmatrix} 12800 & 9600 & -12800 & -9600 \\ 9600 & 7200 & -9600 & -7200 \\ -12800 & -9600 & 12800 & 9600 \\ -9600 & -7200 & 9600 & 7200 \end{bmatrix}$$

Assembling global stiffness matrix:

$$\underline{\underline{K}}=\begin{bmatrix} 25000 & 0 & -25000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -25000 & 0 & 25000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12800 & 9600 & -12800 & -9600 \\ 0 & 0 & 9600 & 7200 & -9600 & -7200 \\ 0 & 0 & -12800 & -9600 & 12800 & 9600 \\ 0 & 0 & -9600 & -7200 & 9600 & 7200 \end{bmatrix}$$

$$\underline{\underline{K}}=\begin{bmatrix} 25000 & 0 & -25000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -25000 & 0 & 25000+12800 & 0+9600 & -12800 & -9600 \\ 0 & 0 & 0+9600 & 0+7200 & -9600 & -7200 \\ 0 & 0 & -12800 & -9600 & 12800 & 9600 \\ 0 & 0 & -9600 & -7200 & 9600 & 7200 \end{bmatrix}$$

Global stiffness matrix:	Global nodal load vector:
$\underline{\underline{K}}=\begin{bmatrix} 25000 & 0 & -25000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -25000 & 0 & 37800 & 9600 & -12800 & -9600 \\ 0 & 0 & 9600 & 7200 & -9600 & -7200 \\ 0 & 0 & -12800 & -9600 & 12800 & 9600 \\ 0 & 0 & -9600 & -7200 & 9600 & 7200 \end{bmatrix}\frac{\mathrm{N}}{\mathrm{mm}}$	$\underline{\underline{f}}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ -F_0 \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ -18000 \\ 0 \\ 0 \end{bmatrix}\mathrm{N}$

The basic $\underline{\underline{K}}\,\underline{q}=\underline{f}$ equation:

$$\begin{bmatrix} 25000 & 0 & -25000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -25000 & 0 & 37800 & 9600 & -12800 & -9600 \\ 0 & 0 & 9600 & 7200 & -9600 & -7200 \\ 0 & 0 & -12800 & -9600 & 12800 & 9600 \\ 0 & 0 & -9600 & -7200 & 9600 & 7200 \end{bmatrix}\cdot\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ -18000 \\ 0 \\ 0 \end{bmatrix}$$

Taking into account the constraints: $q_1=0\ ,q_2=0,q_5=0,q_6=0$

The 1st, 2nd, 5th and 6th rows and columns can be removed:

$$\begin{bmatrix} \cancel{25000} & \cancel{0} & \cancel{-25000} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{-25000} & \cancel{0} & 37800 & 9600 & \cancel{-12800} & \cancel{-9600} \\ \cancel{0} & \cancel{0} & 9600 & 7200 & \cancel{-9600} & \cancel{-7200} \\ \cancel{0} & \cancel{0} & \cancel{-12800} & \cancel{-9600} & \cancel{12800} & \cancel{9600} \\ \cancel{0} & \cancel{0} & \cancel{-9600} & \cancel{-7200} & \cancel{9600} & \cancel{7200} \end{bmatrix}\cdot\begin{bmatrix} \cancel{0} \\ \cancel{0} \\ q_3 \\ q_4 \\ \cancel{0} \\ \cancel{0} \end{bmatrix}=\begin{bmatrix} \cancel{0} \\ \cancel{0} \\ 0 \\ -18000 \\ \cancel{0} \\ \cancel{0} \end{bmatrix}$$

The condensed matrices:

$$\underline{\underline{K}}=\begin{bmatrix} 37800 & 9600 \\ 9600 & 7200 \end{bmatrix}\frac{\mathrm{N}}{\mathrm{mm}}\ ,\quad \underline{\underline{f}}=\begin{bmatrix} 0 \\ -18000 \end{bmatrix}\mathrm{N}$$

The equation system to solve:

$$\underline{\underline{K}}\,\underline{q}=\underline{f}\quad\Rightarrow\underline{q}\,\underline{\underline{K}}^{-1}\underline{f}$$

$$\begin{bmatrix} 37800 & 9600 \\ 9600 & 7200 \end{bmatrix}\cdot\begin{bmatrix} q_3 \\ q_4 \end{bmatrix}=\begin{bmatrix} 0 \\ -18000 \end{bmatrix}$$

$$\mathrm{adj}\left(\underline{\underline{K}}\right)=\begin{bmatrix} 7200 & -9600 \\ -9600 & 37800 \end{bmatrix}$$

$$\det\left(\underline{\underline{K}}\right)=37800\cdot7200-9600\cdot9600=180000000$$

$$\underline{\underline{K}}^{-1}=\frac{\operatorname{adj}\left(\underline{\underline{K}}\right)}{\det\left(\underline{\underline{K}}\right)}=\frac{\begin{bmatrix}7\,200 & -9\,600\\ -9\,600 & 37\,800\end{bmatrix}}{180\,000\,000}=\begin{bmatrix}0.00004 & -0.00005\dot{3}\\ -0.00005\dot{3} & 0.00021\end{bmatrix}$$

$$\begin{bmatrix}q_3\\ q_4\end{bmatrix}=\begin{bmatrix}0.00004 & -0.00005\dot{3}\\ -0.00005\dot{3} & 0.00021\end{bmatrix}\begin{bmatrix}0\\ -18\,000\end{bmatrix}=\begin{bmatrix}0.96\\ -3.78\end{bmatrix}\text{mm}$$

$$\vec{v}_C=\left(0.96\vec{i}-3.78\vec{j}\right)\text{mm}$$

- Nodal forces (loads and reaction forces)

$$\underline{f}=\underline{\underline{K}}\,\underline{q}$$

$$\begin{bmatrix}F_{1x}\\ F_{1y}\\ F_{2x}\\ F_{2y}\\ F_{3x}\\ F_{3y}\end{bmatrix}=\begin{bmatrix}25\,000 & 0 & -25\,000 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ -25\,000 & 0 & 37\,800 & 9\,600 & -12\,800 & -9\,600\\ 0 & 0 & 9\,600 & 7\,200 & -9\,600 & -7\,200\\ 0 & 0 & -12\,800 & -9\,600 & 12\,800 & 9\,600\\ 0 & 0 & -9\,600 & -7\,200 & 9\,600 & 7\,200\end{bmatrix}\cdot\begin{bmatrix}0\\ 0\\ 0.96\\ -3{,}78\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}-24\,000\\ 0\\ 0\\ -18\,000\\ 24\,000\\ 18\,000\end{bmatrix}$$

$$\vec{F}_A=\left(-24\,000\vec{i}+0\vec{j}\right)\text{N}$$

$$\vec{F}_B=\left(24\,000\vec{i}+18\,000\vec{j}\right)\text{N}$$

ELEMENT ①	**ELEMENT ②**
change of length: $\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}c_x & c_y & 0 & 0\\ 0 & 0 & c_x & c_y\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\\ q_4\end{bmatrix}=$ $=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\ 0\\ 0.96\\ -3.78\end{bmatrix}=\begin{bmatrix}0\\ 0.96\end{bmatrix}$ $\Delta L=u_2-u_1=0.96-0=0.96\text{ mm}$	change of length: $\begin{bmatrix}u_3\\ u_2\end{bmatrix}=\begin{bmatrix}c_x & c_y & 0 & 0\\ 0 & 0 & c_x & c_y\end{bmatrix}\begin{bmatrix}q_5\\ q_6\\ q_3\\ q_4\end{bmatrix}=$ $=\begin{bmatrix}0.8 & 0.6 & 0 & 0\\ 0 & 0 & 0.8 & 0.6\end{bmatrix}\begin{bmatrix}0\\ 0\\ 0.96\\ -3.78\end{bmatrix}=\begin{bmatrix}0\\ -1.5\end{bmatrix}$ $\Delta L=u_2-u_3=-1.5-0=-1.5\text{ mm}$
Normal strain: $\varepsilon_\xi=\frac{\Delta L}{L}=\frac{0.96}{800}=0.0012$	Normal strain: $\varepsilon_\xi=\frac{\Delta L}{L}=\frac{-1.5}{1000}=-0.0015$
Normal stress: $\sigma_\xi=E\varepsilon_\xi=$ $=200\,000\cdot 0.0012=240\text{ MPa}$	Normal stress: $\sigma_\xi=E\varepsilon_\xi=$ $=200\,000\cdot(-0.0015)=-300\text{ MPa}$
Normal force: $N=\sigma_\xi A=240\cdot 100=24\,000\text{ N}$	Normal force: $N=\sigma_\xi A=-300\cdot 100=-30\,000\text{ N}$