Választási modellek 2

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2018

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Forrás: A Self Instructing Course in Mode Choice Modeling: Multinomial and Nested Logit Models Prepared For U.S. Department of Transportation Federal Transit Administration By Frank S. oppelman and Chandra Bhat with technical support from Vaneet Sethi, Sriram Subramanian, Vincent Bernardin and Jian Zhang January 31, 2006 Modified June
http://www.caee.utexas.edu/prof/bhat/COURSES/LM_Draft_060131Final-060630.pdf

Preferencia vizsgálatok

- Két alapvető módszer: feltárt preferencia (revealed preference) és kijelentett preferencia (stated preference)
- Feltárt preferencia volt előbb, kijelentett a 80-as évektől
- Feltárt preferencia: tényleges választások
  - Korlátozott számú eset létezik
  - Az attribútumok nem változtathatók
- Kijelentett preferencia (SP): feltételezett (hyphothetical) választások

  „Nem bízunk abban, amit az emberek mondanak, de van olyan helyzet, amikor nem tehetünk mást”

  - SP megbízható, ha a válaszadó érti a feladatot és meg akarja oldani
  - Systematic component-megfigyelhető, random component nem
    - Bármilyen helyzet előállítható
    - Probléma a válaszadók „túlzott lelkesedése”, kezelhető

- Három elterjedt SP módszer van
  - *contingent valuation CV (véletlen, lehetőség)*
  - *conjoint analysis* CA (*egyesített)*
  - *stated choice* SC technika, ez a legelterjedtebb, a közlekedésben is
- CV elsősorban *willingness-to-pay* (WTP) információk beszerzésére alkalmas, a személyi tulajdonságokra is kitejed, az alternatívákat rangsorolni vagy egymáshoz viszonyítva pontozni kell, szokásos több (nem egy-kettő), gyakran számos alternatíva egymáshoz viszonyított értékelése
- CA általános preferencia vizsgálat termékek és szolgáltatások nagyobb számú attribútumainak figyelembe vételével, jellemzően arra kérik a válaszadókat, hogy rangsorolják, vagy értékleják, pontozzák az alternatívákat, ezzel minden alternatívára adnak információt,
- Gyakran több (nem két-három) alternatívát szerepeltetnek (pl. különböző gyártó azonos termékei, vagy hasonló célra szolgáló termékek)

Fare	Interchange	Time on bus	Walk time
70 p	No change	15 mins	10 mins

Fare	Interchange	Time on bus	Walk time
70 p	No change	20 mins	8 mins

Fare	Interchange	Time on bus	Walk time
85 p	No change	15 mins	10 mins

Fare	Interchange	Time on bus	Walk time
85 p	1 change	15 mins	8 mins

Figure 3.6 Example of stated-preference ranking exercise

	Train	Bus		Car
Fare	$3.00	$4.00	Petrol Costs	$1.00
			Toll Cost	$3.00
			Parking Cost	$8.00
Access Time	5 mins	10 mins		
In-vehicle Time	35 mins	25 mins	In-vehicle Time	15 mins
Egress Time	15 mins	10 mins	Egress Time	5 mins
I would choose	○	○	or	○

(a) Standard design

	Train	Bus		Car	None
Fare	$3.00	$4.00	Petrol Costs	$1.00	
			Toll Cost	$3.00	
			Parking Cost	$8.00	
Access Time	5 mins	10 mins			
In-vehicle Time	35 mins	25 mins	In-vehicle Time	15 mins	
Egress Time	15 mins	10 mins	Egress Time	5 mins	
I would choose	○	○	or	○	○

(b) Design including a non-purchase option

Figure 3.7 Example of labelled mode SC tasks

	Route A	Route B	Route C
Petrol Costs	$1.50	$2.00	$1.00
Toll Cost	$2.00	$4.00	$0.00
Prob. of arriving late	0.3	0.5	0.1
Prob. of arriving early	0.1	0.2	0.3
Free Flow Time	15 mins	10 mins	20 mins
Congested Time	10 mins	15 mins	20 mins
Egress Time	15 mins	10 mins	5 mins
Please rank in order of preference (1 = best)			

Figure 3.8 Example of unlabelled route SC task

- SC felvételnél egy kiválasztása történik a lehetőségek közül, nincs rangsorolás, vagy pontozás
- Jellemző, hogy konstruált szituációkban kell választani, amelyet az elemző alkot meg, de ezek hasonlóak való világban előforduló esetekhez
- Csak kevés (legtöbbször csak két) alternatívát mutatnak, ezeket változtatják és egy interjúalany több választ ad
- A válaszolókat arra kérik, úgy válasszanak, ahogy a valóságban is tennék hasonló helyzetben

- The *decision context* may be a hypothetical or a real one; in other words, the respondent may be asked to consider an actual journey or one that she might consider undertaking in the future.
- Some of the *alternatives* offered may be hypothetical although it is recommended that one of them be an existing one, for example the mode just chosen by the respondent including all its attributes
- A crucial problem with stated preference data collection in general, is how much faith we can put on individuals actually doing what they stated they would do when the case arises
- experience in the 1970s was not good in this sense, with large differences between predicted and actual choice
- The situation improved considerably in the 1980s and good agreement with reality was reported from models estimated using SC data (Louviere 1988a).
- However, this occurred because data-collection methods improved enormously and became very demanding, not only in terms of survey design expertise but also in their requirements for trained survey staff and quality-assurance procedures
- the process of constructing effective SP surveys is far from simple and quite time consuming if done correctly.

- A mintát a kiválasztott személyek jelentik- reprezentativitás?
  - A döntést befolyásoló attribútumok szerint kell reprezentatívnak lenni (életkor, foglalkozás, jövedelem, autó tulajdon…)
  - Több attribútum szerint is reprezentatívnak kell lenni, ehhez ismernünk kell az érdekelt lakosság attribútumok szerinti összetételét
  - A mintába kerülők az adott kérdésekben érdekeltek legyenek
- Mintanagyság
  - A modell változóinak száma befolyásolja, minél több változó, annál nagyobb minta
  - ~ 1000-3000 elem, 2/3-ad a modellhez, 1/3-ad a validáláshoz

SP eredmények

	Alternatíva1		Alternatíva2		Választás	
Válasz	Idő1 T1	Költs1 C1	Idő2 T2	Költs2 C2	Alt1	Alt2
1	30	8	25	12	0	1
2	28	18	40	7	1	0
3	26	10	42	6	1	0

2 alternatíva, 2 attribútum, 3 kísérlet

Hasznossági függvény: $V = \beta_1 T + \beta_2 C$

Keressük azokat a bétákat, amelynél a legnagyobb a valószínűsége a választásoknak.

Maximum likelihood becslés

- Elv: egy változó eloszlásában szerepel egy paraméter (α), amelynek értékét nem ismerjük, de rendelkezésünkre áll n elemű minta. A mintaelemek is valószínűségi változók, az α érték valamely lehetséges behelyettesítésével kiszámítható a minta előfordulásának valószínűsége. Az α változtatásával meghatározhatjuk azon értékét, amely mellett a minta előfordulási valószínűsége a legnagyobb.
- Módszer: fel kell állítani a likelihood függvényt, amely a minta egyes elemei előfordulási valószínűségeinek a szorzata. Ez a függvény tartalmazza a keresett α tényezőt, ez lesz a függvény változója. A függvénynek erre a változóra értelmezett maximumát kell meghatározni, és a maximumhoz tartozó α a keresett érték.
- A likelihood függvény a minta elemek számított előfordulási valószínűségeinek a szorzata.

Jelölje i a választást végző személyeket, és q a választható alternatívákat.

$P_{i,\ q}$ = annak valószínűsége, hogy i személy a q alternatívát választja.
A $\prod_q P_{q,i}$ azoknak a valószínűségeknek a szorzata, amellyel i személy az egyes alternatívákat választja. Az össze személy összes alternatívára vonatkozó választási valószínűségeinek szorzata a $\prod_q \prod_i P_{i,q}$, ez a likelihood függvény.

L= $\prod_q \prod_i (P_{i,q})^{f_{i,q}}$

$f_{i,q}$ dummy változó, értéke 1, ha i egyén a q alternatívát választja, ha nem választja, akkor 0.

Ennél a függvénynél a szélsőérték meghatározása nehéz, ezért a logaritmusának a szélsőértékét határozzák meg, tekintettel, hogy a szélsőérték pontok megegyeznek. A loglikelihood függvény a a valószínűségek összege lesz.

LL= $\sum_i \sum_j f_{i,q} ln P_{i,q}$

Példa likelihood függvényre

$$V_{Auto} = \beta_1 \times Travel\ \ Time_{Auto}$$
$$V_{Bus} = \beta_1 \times Travel\ \ Time_{Bus}$$

Individual #	**Auto Travel Time**	**Bus Travel Time**	**Chosen Mode**
1	30 minutes	50 minutes	Car (mode 1)
2	20 minutes	10 minutes	Car (mode 1)
3	40 minutes	30 minutes	Bus (mode 2)

Két alternatíva, egy attribútum. Egy attribútum mellett valószínűségi döntés az ismeretlen (random) tényező léte miatt lehetséges.
A választások valószínűségei:

$$Individual\ 1\ (P_{11}) = \frac{\exp(30\beta)}{\exp(50\beta) + \exp(30\beta)} = \frac{1}{1+\exp(20\beta)}$$

$$Individual\ 2\ (P_{12}) = \frac{\exp(20\beta)}{\exp(10\beta) + \exp(20\beta)} = \frac{1}{1+\exp(-10\beta)}$$

$$Individual\ 3\ (P_{23}) = \frac{\exp(30\beta)}{\exp(30\beta) + \exp(40\beta)} = \frac{1}{1+\exp(10\beta)}$$

A likelihood függvény

$$L = \frac{1}{1+\exp(20\beta)} \times \frac{1}{1+\exp(-10\beta)} \times \frac{1}{1+\exp(10\beta)}$$

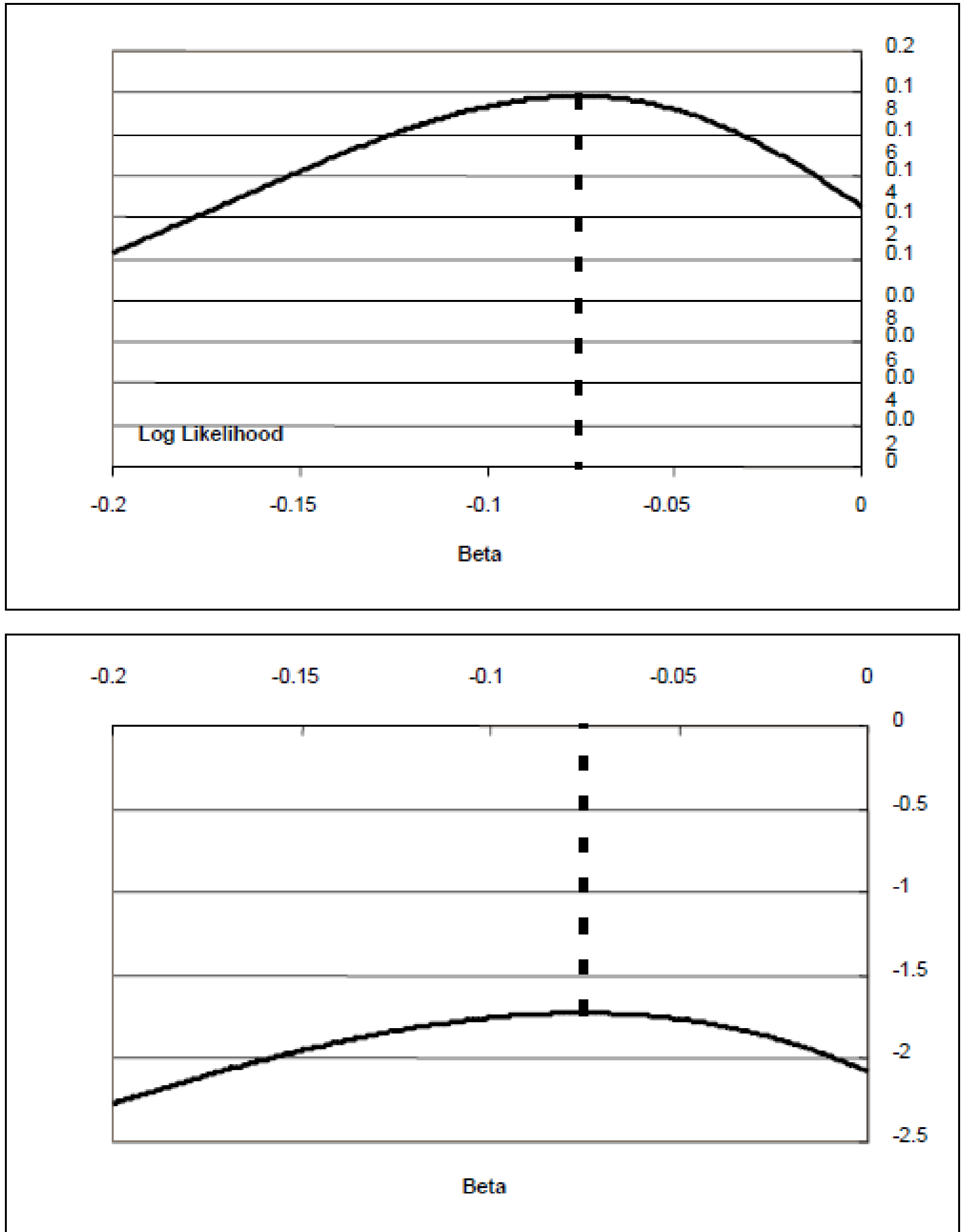
A loglikelihood függvény

$$LL = \ln\left(\frac{1}{1+\exp(20\beta)}\right) + \ln\left(\frac{1}{1+\exp(-10\beta)}\right) + \ln\left(\frac{1}{1+\exp(10\beta)}\right)$$

Derivált

$$0 = -\left(\frac{20e^{20\beta}}{e^{20\beta}+1}\right) + \left(\frac{10}{e^{10\beta}+1}\right) - \left(\frac{10e^{10\beta}}{e^{10\beta}+1}\right)$$

β=-0,075630761



ıre 4.7 Likelihood and Log-likelihood as a Function of a Parameter Value

A logit modell másik formája

$$\Pr(DA) = \frac{\exp(V_{DA})}{\exp(V_{DA}) + \exp(V_{SR}) + \exp(V_{TR})} \times \frac{\exp(-V_{DA})}{\exp(-V_{DA})}$$

$$= \frac{\exp(V_{DA}) \times \exp(-V_{DA})}{\left[\exp(V_{DA}) + \exp(V_{SR}) + \exp(V_{TR})\right] \times \exp(-V_{DA})}$$

$$= \frac{\exp(0)}{\exp(0) + \exp\left(V_{SR} - V_{DA}\right) + \exp\left(V_{TR} - V_{DA}\right)}$$

$$\Pr(DA) = \frac{1}{1 + \exp\left(V_{SR} - V_{DA}\right) + \exp\left(V_{TR} - V_{DA}\right)}$$

$$\Pr(i) = \frac{1}{1 + \sum_{j \neq i} \exp(V_j - V_i)} \qquad \forall i \in J$$

Példa2

Két alternatíva, két attribútum, három válasz

	Alternatíva1		Alternatíva2		Választás	
Válasz	Idő T1	Költség C1	Idő T2	Költség C2	Alt1	Alt2
1	30	8	25	12	0	1
2	28	18	40	7	1	0
3	26	10	42	6	1	0

V= $\beta_1 T + \beta_2 C$

Likelihood függvény

$$L(\beta_1, \beta_2) =$$

$$\{1/[1 + exp((25\beta_1 + 12\beta_2) - (30\beta_1 + 8\beta_2))]\}^0 \text{ x}$$

$$\{1/[1 + exp((30\beta_1 + 8\beta_2) - (25\beta_1 + 12\beta_2))]\}^1 \text{ x}$$

$$\{1/[1 + exp((40\beta_1 + 7\beta_2) - (28\beta_1 + 18\beta_2))]\}^1 \text{ x}$$

$$\{1/[1 + exp((28\beta_1 + 18\beta_2) - (40\beta_1 + 7\beta_2))]\}^0 \text{ x}$$

$$\{1/[1 + exp((42\beta_1 + 6\beta_2) - (26\beta_1 + 10\beta_2))]\}^1 \text{ x}$$

$$\{1/[1 + exp((26\beta_1 + 10\beta_2) - (42\beta_1 + 6\beta_2))]\}^0$$

loglikelihood függvény

$$LL(\beta_1, \beta_2) =¶$$

$$ln\{1/[1 + exp((30\beta_1 + 8\beta_2) - (25\beta_1 + 12\beta_2))]\} +¶$$

$$\ln\{1/[1 + exp((40\beta_1 + 7\beta_2) - (28\beta_1 + 18\beta_2))]\} +¶$$

$$\ln\{1/[1 + exp((42\beta_1 + 6\beta_2) - (26\beta_1 + 10\beta_2))]\}¶$$

$$LL(\beta_1, \beta_2) = \ln[1/(1 + exp(5\beta_1 - 4\beta_2)] +$$

$$\ln[1/(1 + exp(12\beta_1 - 11\beta_2)] + \ln[1/(1 + exp(16\beta_1 - 4\beta_2)]$$

Biogeme data file

In order to estimate the coefficients of the model, we use a data file called `sample.dat` including (among others) the following data: `Choice`, taking value from 1 to 6 and identifying the choice actually made for each observation, `x11`, `x21`, `x31`, `x41`, `x51`, `x61`, `x12`, `x22`, `x32`, `x42`, `x52`, `x62`, which are

the two variables of the model associated with each alternative, `av1`, `av2`, `av3`, `av4`, `av5`, `av6`, which have the value 1 if the associated alternative is available for the current observation, and 0 otherwise.

BIOGEME 1.8 assumes that the data file contains in its first line a list of labels corresponding to the available data, and that each subsequent line contains the exact same number of numerical data, each row corresponding to an observation. Delimiters can be tabs or spaces.

Biogeme

BIOGEME 1.8 also needs a file where the model specification is described. Use your favorite text editor to create such a file, which must have `.mod` as an extension. This file is organized into many sections, most of them being optional. For our simple example, we must define at least 5 sections:

1. Section `Choice`: it simply describes to BIOGEME 1.8 where the dependent variable can be found in the file. The syntax is

   ```
   [Choice]
   Choice
   ```

   Note that the syntax is case sensitive, and that `choice` is different from `Choice`.

2. Section Beta: this section describes to BIOGEME 1.8 the list of coefficients that must be estimated. The syntax is

```
[Beta]
// Name Value  LowerBound UpperBound  status (0=variable, 1=fi:
ASC1      0         -10000         10000       1
ASC2      0         -10000         10000       0
ASC3      0         -10000         10000       0
ASC4      0         -10000         10000       0
ASC5      0         -10000         10000       0
ASC6      0         -10000         10000       0
BETA1         0         -10000         10000       0
BETA2         0         -10000         10000       0
```

We immediately note that every line starting by `//` are ignored by BIOGEME 1.8 and are used to include comments in the file. The section is organized into columns. The first column contains the name of the coefficients. The second column contains a default value (usually 0). The third and fourth columns contain lower and upper bounds (respectively) on the value of the coefficients. Using -10000 and 10000 by default is appropriate is the vast majority of practical cases. The last column tells BIOGEME 1.8 is the coefficient must be estimated (0), or must be maintained at its default value (1). In this example, not all alternative specific constants are identified. Therefore, `ASC1` is fixed to its default value 0 by putting a 1 in the last column.

3. Section `Utilities`: This is were the specification of the utility functions is described. The specification for one alternative must start at a new row, and may actually span several rows. For each of them, four entries are specified:

   (a) The identifier of the alternative, with a numbering convention consistent with section [Choice];

   (b) The name of the alternative;

   (c) The availability condition. In this case, it is a direct reference to one of the entries in the data file;

   (d) The linear-in-parameter utility function is composed of a list of terms, separated by a `+`. Each term is composed of the name of a parameter and the name of an attribute, separated by a `*`. Note that a space is required after each parameter name parameter.

```
[Utilities]
// Id Name  Avail  linear-in-parameter expression
  1   Alt1   av1   ASC1 * one + BETA1 * x11 + BETA2 * x12
  2   Alt2   av2   ASC2 * one + BETA1 * x21 + BETA2 * x22
  3   Alt3   av3   ASC3 * one + BETA1 * x31 + BETA2 * x32
  4   Alt4   av4   ASC4 * one + BETA1 * x41 + BETA2 * x42
  5   Alt5   av5   ASC5 * one + BETA1 * x51 + BETA2 * x52
  6   Alt6   av6   ASC6 * one + BETA1 * x61 + BETA2 * x62
```

4. Section `Expressions`: it describes to BIOGEME 1.8 how to compute attributes not directly available from the data file. In this example, the only such attribute is `one`.

```
[Expressions]

// Define here arithmetic expressions for name that are not
// directly available from the data
one = 1
```

5. Section `Model`: it tells BIOGEME 1.8 which assumptions must be used regarding the error term, that is which type of model must be estimated. In this example, it is the multinomial logit model.

```
[Model]
// Currently, only $MNL (multinomial logit),
// $NL (nested logit), $CNL (cross-nested logit) and
// $NGEV (Network GEV model) are valid keywords
$MNL
```

Assume we have typed these sections in the file `mymodel.mod`. BIOGEME 1.8 is then run in a shell using the command

```
biogeme mymodel sample.dat
```

The following information appears on the screen:

- Information about the version of BIOGEME. The date is when the software was compiled.

  ```
  BIOGEME Version 1.7 [Sun Aug 3 11:04:51 2008]
  Michel Bierlaire, EPFL
  ```

- BIOGEME 1.8 checks is a file called `mymodel.par`, containing various parameters, exists. If not, it checks if the file called `default.par` exists. If not, it creates it and set default values to the parameters. That's what most users need in the beginning. Note that the information like `[14:42:32]patFileNames.cc:68` can be safely ignored.

  ```
  [14:42:32]patFileNames.cc:68  mymodel.par does not exist
  [14:42:32]patFileNames.cc:72  Trying default.par instead
  [14:42:32]patBiogeme.cc:135  File default.par does not exist. Default values
  [14:42:32]patBiogeme.cc:137  A file default.par has been created
  ```

- BIOGEME 1.8 then reads the model and data files and reports various information.

```
Read headers in sample.dat
22 headers read in sample.dat
Total number of different headers: 22
Read file mymodel.mod
Opening file sample.dat
Data  file... line 500 Memory: 131 Kb
Data  file... line 1000 Memory: 263 Kb
Total obs.:   999
Total memory: 263.672 Kb
Detailed info in mymodel.sta
Nbr of attributes: 23
Nbr of alternatives: 6
Sample size: 1000
Nbr of groups: 1
Nbr of betas: 8
Run time for data processing: 00:01
Dimension of the optimisation problem = 7
Utility function: Linear-in-parameters
```

- BIOGEME 1.8 then starts the estimation. It displays miscellaneous information at each iteration of the estimation algorithm.

```
Init loglike=-1605.17
    gmax Iter   radius        f(x)     Status       rhok nFree
+1.47e+00    1 1.00e+00 +1.6051696e+03 ****Converg  +1.14e+00 7   ++
+3.61e-01    2 2.00e+00 +7.9759926e+02 ****Converg  +1.24e+00 7   ++
+1.27e-01    3 4.00e+00 +6.2840108e+02 ****Converg  +1.20e+00 7   ++
+3.62e-02    4 8.00e+00 +5.8370521e+02 ****Converg  +1.11e+00 7   ++
+5.55e-03    5 1.60e+01 +5.7697041e+02 ****Converg  +1.01e+00 7   ++
+4.50e-04    6 3.20e+01 +5.7600869e+02 ****Converg  +1.00e+00 7   ++
+3.97e-05    7 6.40e+01 +5.7600213e+02 ****Converg  +1.00e+00 7   ++
Convergence reached...
```

- BIOGEME 1.8 reports the running time and prepares the output files.

```
Run time: 00:01
Final log-likelihood=-576.002
Be patient... BIOGEME is preparing the output files
Run time for var/covar computation: 00:00
```

- For the record, BIOGEME 1.8 reports the list of files that were actually used as input.

```
BIOGEME Input files
===================
Parameters: default.par
Model specification: mymodel.mod
Sample 1 : sample.dat
```

- BIOGEME 1.8 reports the list of files that have been created, containing the results of the estimation, as well as many other pieces of information.

```
BIOGEME Output files
====================
Estimation results: mymodel.rep
Estimation results (HTML): mymodel.html
Estimation results (Latex): mymodel.tex
Estimation results (ALogit): mymodel.F12
Result model spec. file: mymodel.res
Sample statistics: mymodel.sta
```

- BIOGEME 1.8 reports also the name of files that may be helpful in understanding problems with the model.

```
BIOGEME Debug files
===================
Log file: mymodel.log

Parameters debug: parameters.out
Model debug: model.debug
Model spec. file debug: __specFile.debug
```

- BIOGEME 1.8 reports some information specific to the model. For MNL, it reports the minimum argument of all exponentials computed during the process, in order to signal a possible underflow. Most users do not worry about this information.

```
Model informations: Multinomial Logit Model
==================
The minimum argument of exp was -18.2387
```

- Finally, BIOGEME 1.8 reports the time of the run.

```
Run time for estimation:      00:02
Total run time:               00:03
```

For the results, most users will consult the HTML file using their preferred browser. A file written in ASCII format is also available, with the extension `.rep`. A file with LaTeX code is also created, so that the results can easily be integrated in a report or an article written with this word processor. The output looks like:

Model	:	Multinomial Logit
Number of estimated parameters	:	7
Number of observations	:	1000
Number of individuals	:	1000
Null log-likelihood	:	-1605.170
Init log-likelihood	:	-1605.170
Final log-likelihood	:	-576.002
Likelihood ratio test	:	2058.335
Rho-square	:	0.641
Adjusted rho-square	:	0.637
Final gradient norm	:	+1.150e-04
Diagnostic	:	Convergence reached...
Iteration	:	7
Run time	:	00:01
Variance-covariance	:	from analytical hessian
Sample file	:	sample.dat

Summary statistics		
Number of observations = 1000		
$\mathcal{L}(0)$	=	-1605.170
$\mathcal{L}(\hat{\beta})$	=	-576.002
$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})]$	=	2058.335
ρ^2	=	0.641
$\bar{\rho}^2$	=	0.637

Variable number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	ASC2	-0.168	0.164	-1.02	0.31
2	ASC3	-0.0437	0.172	-0.25	0.80
3	ASC4	-0.486	0.180	-2.70	0.01
4	ASC5	0.703	0.167	4.20	0.00
5	ASC6	-1.34	0.207	-6.48	0.00
6	BETA1	0.759	0.0372	20.43	0.00
7	BETA2	0.776	0.0368	21.06	0.00

- `Init log-likelihood` is the log-likelihood of the sample for the model defined in the `.mod` file.
- `Final log-likelihood` is the log-likelihood of the sample for the estimated model.
- `Likelihood ratio test` is

$$-2(\mathcal{L}^0 - \mathcal{L}^*) \tag{5}$$

where $\mathcal{L}^0$ is the log-likelihood of the sample for a Multinomial Logit model where all β parameters are 0, defined by (3), and $\mathcal{L}^*$ is the log-likelihood of the sample for the estimated model.

- `Rho-square` is

$$\rho^2 = 1 - \frac{\mathcal{L}^*}{\mathcal{L}^0}. \tag{6}$$

- `Adjusted rho-square` is

$$\rho^2 = 1 - \frac{\mathcal{L}^* - K}{\mathcal{L}^0}. \tag{7}$$

where K is the number of estimated parameters. Note that this statistic is meaningless in the presence of constraints, where the number of degrees of freedom is less than the number of parameters.

- `Final gradient norm` is the gradient of the log-likelihood function computed for the estimated parameters. If no constraint is active at the solution, it should be close to 0. If there are equality constraints, or if some bound constraints or inequality constraints are active at the solution (that is, they are verified with equality), the gradient may not be close to zero.

- `Null log-likelihood` is the log-likelihood of the sample for a Multinomial Logit model where all β parameters are 0. It is computed as

$$\mathcal{L}^0 = \sum_{n \in \text{sample}} \omega_n \ln \frac{1}{\#\mathcal{C}_n} \qquad (3)$$

where $\#\mathcal{C}_n$ is the number of alternatives available to individual n and ω_n is the associated weight.