1. Sets

What is a set?

By sets we mean collections of some objects. Set is not defined mathematically; the notion of set is primitive.

Sets are usually noted by capital letters: A, B, C, …, H, …, S, …

Membership relation $\in$ is also primitive. For elements mostly lower-case letters are used, however they can be sets or other mathematical object (e.g. functions) in those cases we apply the corresponding notations.

$a \in A$ means that "a is a member of set A" "a belongs to A", "a is an element of set A". Its negotiation $a \notin A$ "a is not an element of A".

How to define a set?

There are two ways to define a particular set.

1. Listing all of its elements between set builder braces
   e.g. $A = \{3,4,5,6,7,8\}$ or $B = \{pear, apple, mango\}$.

   Note, that the order of the elements does not affect the set and each element is included only once.

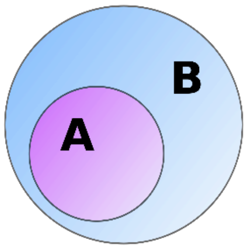
2. Elements are given by a rule. Elements satisfying a given property belongs to the set.
   e.g. $A = \{\text{positive integer numbers greater than } 5\} = \{x \in \mathbb{N} : x > 5\}$;
   $B = \{\text{children born after } 2015\}$.

Two sets are equal if and only if both have the same elements.

Subsets

If each element of A also belongs to B then A is a subset of B. If B may have other additional elements but can be equal, we use the notation $A \subseteq B$. If equality is not allowed, A is a proper subset of B: $A \subset B$.

Venn diagrams are used to represent relations of sets (see Fig. 1.).



1. *Figure: A is a subset of B*

Cardinality

The cardinality of a set is the number of elements of the set.

Empty set has no elements and notated by $\varnothing$. However, $\varnothing \neq \{\varnothing\}$ as the later has one element.

Finite sets have finite number of element (we could in principle count them and finish counting). If we could (at least is principle) count all the elements of the set but counting would never end, the set is denumerably infinite, also called countably infinite. A set is said to be countable whether it is finite or countably infinite.

Common number sets

$\mathbb{N}$ natural numbers $\mathbb{N}=\{1,2,3,4,...\}$

$\mathbb{Z}$ integer numbers $\mathbb{Z}=\{...-4,-3,-2,-1,0,1,2,3,4,...\}$

$\mathbb{Q}$ rational numbers $\mathbb{Q}=\left\{\frac{a}{b}:a,b\in\mathbb{Z};b\neq 0\right\}$

e.g. $4=\frac{4}{1};2.47=\frac{247}{100};2.\dot{3}=\frac{7}{3}$. Rational numbers' decimal form is either final or recurring.

$\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are denumerably infinite.

$\mathbb{Q}^*$ irrational numbers cannot be expressed as fractional numbers and their decimal form has infinite decimal digit with no repetition. E.g. $3.505005000500005...,\pi,\sqrt[3]{5}$

$\mathbb{Q}^*$ is uncountable.

$\mathbb{R}$ real numbers are $\mathbb{Q}$ and $\mathbb{Q}^*$ together.

Operations on sets

1. Union

The union of two sets is defined as the set all the elements that belong to at least one of the sets. Notation: $A\cup B$

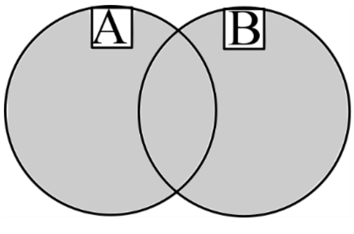
$A\cup A=A$

$A\cup\emptyset=A$

$A\cup B=B\cup A$ i. e. union operation is commutative.

It can be generalized for any number of sets as it has the associative property.

$(A\cup B)\cup C=A\cup(B\cup C)=A\cup B\cup C$



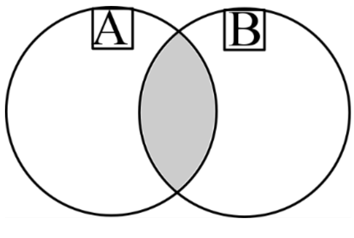
2. Figure: Venn diagram of the union of two sets

2. Intersection

The intersection of two sets contains only those elements that belong to both sets. Notation: $A\cap B$

$A\cap A=A$

$A\cap\emptyset=\emptyset$



3. Figure: Venn diagram of intersection of sets A and B.

$A \cap B = B \cap A$ commutative law.

$(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$ associative law.

Distributive laws

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ union distributes over intersection and

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ intersection distributes over union.

✍ Prove the distributive laws with Venn diagrams.

3. Difference

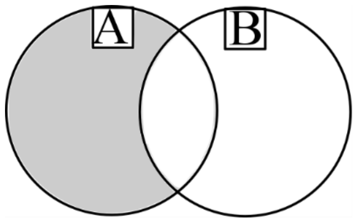
The difference of A and B consists of those elements of *A* which are not present in *B*.

Notation: $A \setminus B$

$A \setminus A = \varnothing$

$A \setminus \varnothing = A$

Note that difference is not commutative $A \setminus B \neq B \setminus A$.

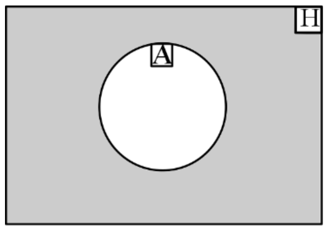


4. Figure A\B represented by a Venn diagram

4. Complement

If A is a subset of *H*, complement of *A* corresponding to H consist of those elements of *H* which are not members of *A*.

$\overline{A} = H \setminus A$



5. Figure: Venn diagram of fcomplement of A in H

De Morgan's laws

Let $A, B \subseteq U$ then the following equations hold:

$\overline{A \cup B} = \overline{A} \cap \overline{B}$ and

$\overline{A \cap B} = \overline{A} \cup \overline{B}$.

✍ Use Venn diagrams to prove the statements.

Solved Problems

Example 1.

If $A = \{1, 2, 3, 4, 10, 11, 12, 16\}$, $B = \{1, 3, 5, 6, 7, 11, 12\}$ and $C = \{1, 2, 5, 8, 9, 12, 15\}$. Find $(A \cap B) \setminus C$.

Solution

At first, we determine $A \cap B$. Intersection of two sets contains those elements which belong to both sets, that means $A \cap B = \{1, 3, 11, 12\}$. When we apply subtraction by deleting all elements from $A \cap B$, which are also elements of set C i.e. 1 and 12. So that $(A \cap B) \setminus C = \{3, 11\}$.

Example 2.

Given $A = \{x \in \mathbb{Z} : x^3 - 25x = 0\}$ and $B = \{x \in \mathbb{N} : |2x - 11| \le 4\}$ find the sets $A \cup B, A \cap B, A \setminus B$ and $B \setminus A$.

Solution

Starting with determining set *A*, we solve the equation $x^3 - 25x = 0$ on the set of integer numbers.

After factoring out common factors we get $x(x^2 - 25) = 0$. It can hold if and only if $x = 0$ or $x = \pm 5$. As all the solutions are integers $A = \{-5, 0, 5\}$.

We can find *B* by solving the inequality $|2x - 11| \le 4$ in the natural number set. As the absolute value of the expression $2x - 11$ is smaller than 4, that means its distance from the origin is not greater than 4.

$-4 \le 2x - 11 \le 4 \qquad /+11$ (for all the three sides)

$7 \le 2x \le 15 \qquad /:2$

$3.5 \le x \le 7.5$

Remember that *B* is a subset of $\mathbb{N}$ so $B = \{4, 5, 6, 7\}$

Set $A \cup B$ contains those elements which belong to any of the two sets: $A \cup B = \{-5, 0, 4, 5, 6, 7\}$.

The intersection of A and B consists of those elements which are members of both sets. Now we have only one common element: $A \cap B = \{5\}$.

Set $A \setminus B$ contains only those elements of *A*, which are absent in *B*. $A \setminus B = \{-5, 0\}$

Set $B \setminus A$ contains only those elements of *B*, which are not in *B*. $B \setminus A = \{4, 6, 7\}$

Example 3.

Let $A = \left\{x \in \mathbb{N} : \frac{2x - 8}{5} \le 2\right\}$, $B = \{x \in \mathbb{Z} : |x - 9| = 5\}$ and $C = \{x \in \mathbb{Z} : 7 - 3x < -8\}$. Find $(A \setminus C) \cup B$.

Solution

At first, we determine set *A*.

$$\frac{2x-8}{5} \le 2 \quad / \cdot 5$$

$$2x-8 \le 10 \quad /+8$$

$$2x \le 18 \quad /:2$$

$$x \le 9$$

As A is a subset of the natural numbers: $A = \{1,2,3,4,5,6,7,8,9\}$.

Example 4.

If $A = \{x \in \mathbb{R} : x^2 - x - 12 \le 0\}$ and $B = \{x \in \mathbb{R} : |2x+3| > 7\}$. Find $A \cup B, A \cap B, A \setminus B$ and B\A.

Solution

First, in order to solve the second order inequality $x^2 - x - 12 \le 0$ we investigate the function $f(x) = x^2 - x - 12$ and determine its x intercepts. We solve the equation $0 = x^2 - x - 12$ by using the formula: $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$x_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-12)}}{2 \cdot 1}$, that is $x_1 = -3$ and $x_2 = 4$.

Now we can plot the graph of the function. As the quadratic coefficient is positive, we have an upward parabola.

The equation $x^2 - x - 12 \le 0$ is true at the interval $A = [-3, 4]$.

For set be we solve the inequality

$|2x+3| > 7$.

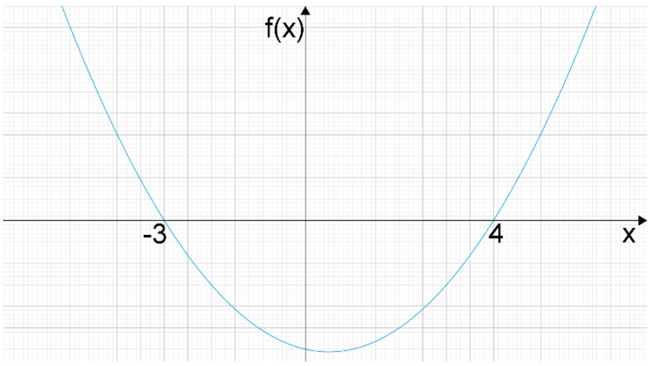
That means:

$2x+3 < -7$ or $2x+3 > 7$

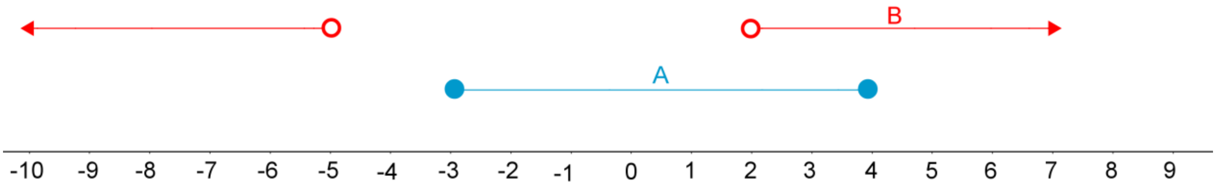
$2x < -10$ $\qquad$ $2x > 4$

$x < -5$ $\qquad$ $x > 2$

Consequently $B =]-\infty, -5[\cup]2, \infty[$.



6. Figure: Graph of the function



7. Figure Sets A and B plotted on the number line

For $A \cup B$ we take those elements which belong to any of the sets, so $A \cup B =]-\infty, -5[\cup [-3, \infty[$.

Set $A \cap B$ contains those elements which belong to both sets, so: $A \cap B =]2, 4]$.

Recall that $A \setminus B$ means the subset of only those elements of A which are not members of B: $A \setminus B = [-3, 2]$.

For the same reason $B \setminus A =]-\infty, -5[\cup]4, \infty[$.

Example 5.

Find the complement of set $A = \{x \in \mathbb{R} : -x^2 - x + 20 > 0\}$ in $\mathbb{R}$.

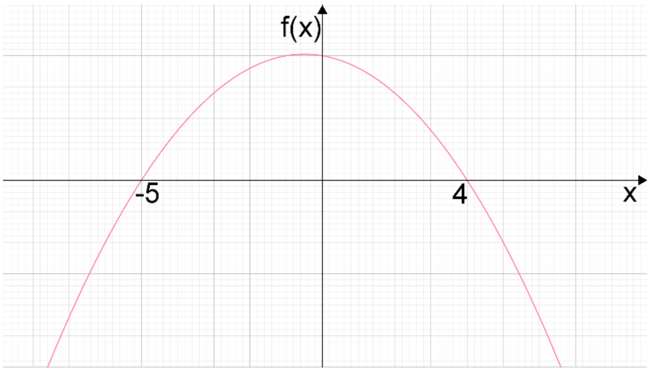
Solution

To find set A we solve the inequality $-x^2 - x + 20 > 0$. We use the general solution of quadratic equations for the function $f(x) = -x^2 - x + 20$.

$$x_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot (-1) \cdot 20}}{2 \cdot (-1)}$$

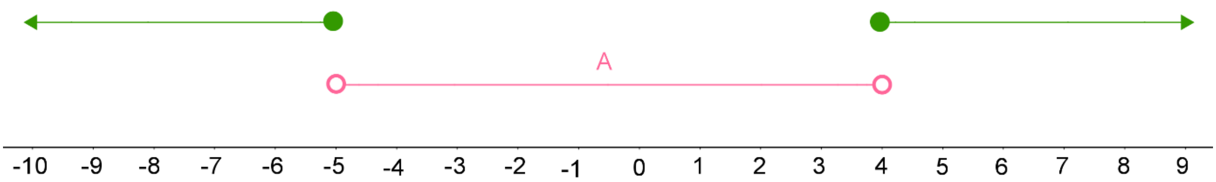
So, we gain the x intercepts -5 and 4.

As the quadratic coefficient is negative the graph of the function is a downward parabola. Since the solutions of the inequality are those real numbers for which the value of the function is over the x axis, $A =]-5, 4[$.



8. Figure Graph of the funcion

Complement of A is $\overline{A} = R \setminus A$ that is $\overline{A} =]-\infty, \infty[\setminus]-5, 4[=]-\infty, -5] \cup [4, \infty[$.



Example 6.

Given $A = \{x \in \mathbb{R} : x^2 - 3x - 4 \leq 0\}$ and $B = \left\{x \in \mathbb{R} : 8 - \frac{5x+7}{2} \leq -3\right\}$. Find the complement of $A \cup B$ in $\mathbb{R}$.

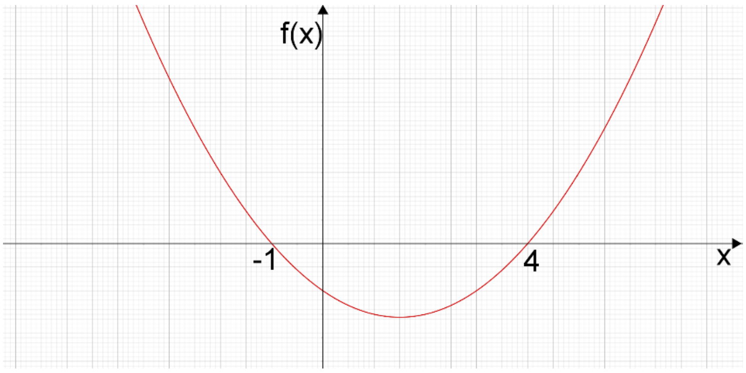
Solution

Set A is a subset of $\mathbb{R}$, for that the inequality $x^2-3x-4<0$ holds. We take the function $f(x)=x^2-3x-4$ and determine its x intersepts:

$$x_{1,2}=\frac{-(-3)\pm\sqrt{(-3)^2-4\cdot1\cdot(-4)}}{2\cdot1}$$

The roots are -1 and 4.

Elements of set A are those real numbers where the value of the function is negative (where the funcion's graph is under the x-axis). $A=[-1,4]$



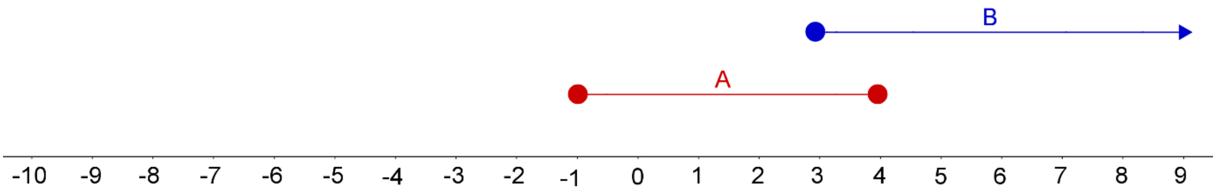
10. *Figure: Graph of f*

For set B we solve the inequality

$$\begin{aligned} 8-\frac{5x+7}{2}&\le -3\\ 16-5x-7&\le -6\\ -5x+9&\le -6\\ -5x&\le -15\\ x&\ge 3 \end{aligned}$$

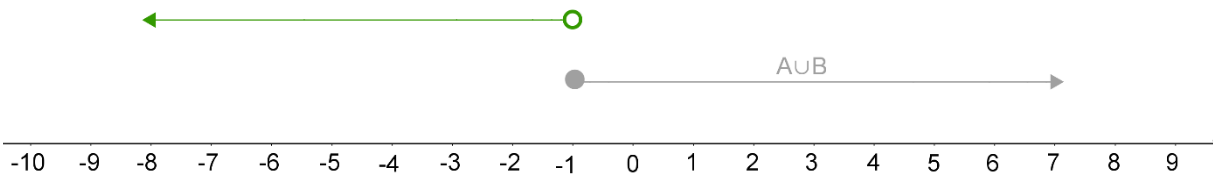
and we get the interval $B=[3,\infty[$.

We plot both sets on number line to get their union.



11. *Figure Sets A and B as intervals*

$A\cup B=[-1,\infty[$ $\qquad \overline{A\cup B}=R\setminus(A\cup B)=]-\infty,\infty[\setminus[-1,\infty[=]-\infty,-1[$



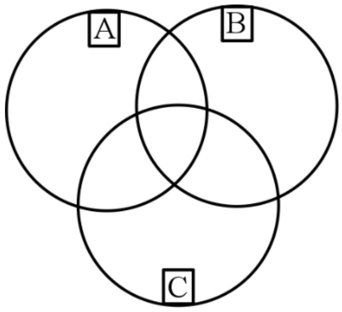
12. *Figure: Finding the complement of* $A\cup B$

Example 7.

Draw a Venn diagram illustrating the set $(B \setminus C) \cup (A \cap B \cap C)$.

Solution

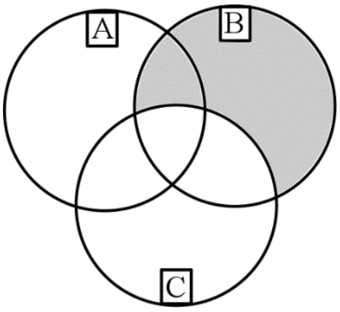
We suppose that the three sets are arbitrary (may have common element and also some elements belong to only one of them).



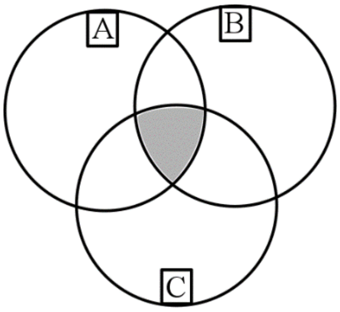
13. *Figure Venn diagram of three sets*

We draw first a representation of set $B \setminus C$. It contains only those elements of B which are not elements of C.

Then we plot $A \cap B \cap C$ containing elements which belong to all the three sets.

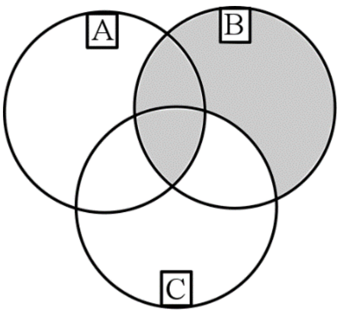


15. *Figure B\C*



14. *Figure: Union of the three sets*

Finally, we take the union of these two sets to get $(B \setminus C) \cup (A \cap B \cap C)$.



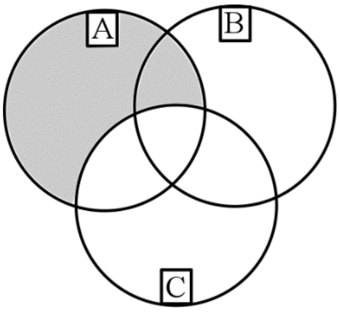
16. *Figure: Solution*

Example 8.

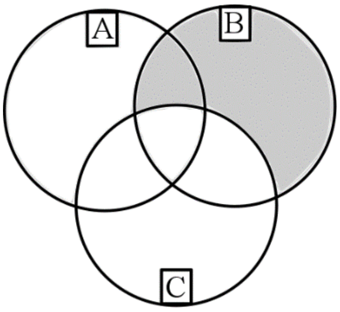
Use Venn diagrams to prove the equation $(A \setminus C) \cap (B \setminus C) = (A \cap B) \setminus C$.

Solution

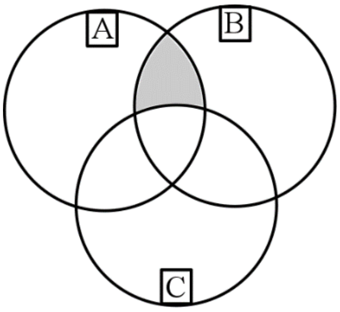
We plot the step by step the LHS then the RHS of the equation and compare them.



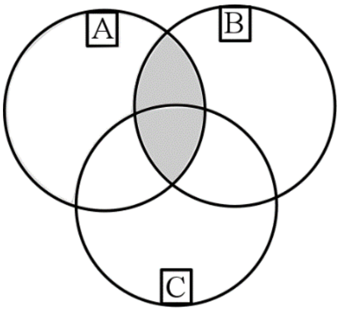
$A \setminus C$



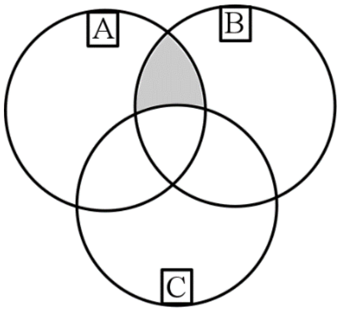
$B \setminus C$



$(A \setminus C) \cap (B \setminus C)$ LHS



$A \cap B$



$(A \cap B) \setminus C$ RHS

LHS=RHS.