1. Gyakorlat

2023. 02. 22.

1. Oldjuk meg Gauss- (Gauss-Jordan) eliminációval az alábbi lineáris egyenletrendszereket!

a)

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ 2x_1 - x_2 + x_3 &= 1 \\ x_1 + 5x_2 + 3x_3 &= -2 \end{aligned}$$

b)

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 3 \\ 2x_1 - x_2 + 5x_3 &= -1 \\ 3x_1 - 3x_2 + 12x_3 &= 2 \end{aligned}$$

c)

$$\begin{aligned} 2x_1 - 3x_2 + x_3 &= -2 \\ 4x_1 - x_2 + x_3 &= 4 \\ 5x_1 - 5x_2 + 2x_3 &= -1 \end{aligned}$$

d)

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - x_4 &= 6 \\ 3x_1 + x_2 - 5x_3 - x_4 &= -8 \\ 2x_1 + x_2 - 4x_3 + 4x_4 &= -2 \\ 3x_1 - 3x_3 + x_4 &= 3 \end{aligned}$$

2. Határozzuk meg az alábbi mátrixok LU-felbontását! Számítsuk ki a mátrixok determinánsának értékét az LU-felbontás segítségével!

a)

$$A = \begin{pmatrix} 2 & 6 & 3 \\ 1 & 5 & 15 \\ 1 & 1 & -2 \end{pmatrix}$$

b)

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

c)

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 6 & 9 \\ -\frac{2}{3} & \frac{2}{3} & 9 \end{pmatrix}$$

d)

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 4 & -1 & 6 \\ 6 & -7 & 8 \end{pmatrix}$$

e)

$$A = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

f)

$$A = \begin{pmatrix} 4 & 3 & 0 & 0 & 0 \\ 2 & 4 & 2 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -1 & 4 \end{pmatrix}$$

3. Oldjuk meg az alábbi egyenletrendszereket LU-felbontással!

a)

$$\begin{aligned} 2x_1 - x_2 + 2x_3 &= 6 \\ 4x_1 - x_2 + 6x_3 &= 20 \\ 6x_1 - 7x_2 + 8x_3 &= 16 \end{aligned}$$

b)

$$\begin{aligned} 2x_1 + 6x_2 - 3x_3 &= -5 \\ x_1 + 5x_2 + 15x_3 &= 14 \\ 3x_1 + 10x_2 + 4x_3 &= 1 \end{aligned}$$

① Oldjuk meg az alábbi egyenletrendszereket Gauss-(Jordan) eliminációval!

b) $x_1 + x_2 - 2x_3 = 3$

$2x_1 - x_2 + 5x_3 = -1$

$3x_1 - 3x_2 + 12x_3 = 2$

Mo:

$$A = \begin{pmatrix} 1 & 1 & -2 \\ 2 & -1 & 5 \\ 3 & -3 & 12 \end{pmatrix}_{3\times 3} ; \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{3\times 1} ; \quad \underline{b} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}_{3\times 1}$$

$$A\,\underline{x} = \underline{b}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 3 \\ 2 & -1 & 5 & -1 \\ 3 & -3 & 12 & 2 \end{array}\right) \begin{array}{l} \\ -2\cdot\mathrm{I} \\ -3\cdot\mathrm{I} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & \boxed{-3} & 9 & -7 \\ 0 & -6 & 18 & -7 \end{array}\right) \begin{array}{l} \\ :(-3) \\ \\ \end{array}$$

(⇒ nuis no.)

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & \boxed{1} & -3 & \frac{7}{3} \\ 0 & -6 & 18 & -7 \end{array}\right) \begin{array}{l} -\mathrm{II} \\ \\ +6\cdot\mathrm{II} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & \frac{2}{3} \\ 0 & 1 & -3 & \frac{7}{3} \\ 0 & 0 & 0 & 7 \end{array}\right)$$

↯ ⇒ nuis no.

c) $2x_1 - 4x_2 + x_3 = -2$

$3x_1 + x_2 - 2x_3 = -3$

$\rightarrow -4x_1 + 2x_2 + x_3 = 4$

Mo :

$$\left(\begin{array}{ccc|c} \boxed{2} & -4 & 1 & -2 \\ 3 & 1 & -2 & -3 \\ -4 & 2 & 1 & 4 \end{array}\right) :2$$

$$\left(\begin{array}{ccc|c} \boxed{1} & -2 & \frac{1}{2} & -1 \\ 3 & 1 & -2 & -3 \\ -4 & 2 & 1 & 4 \end{array}\right) \begin{array}{l} \\ -3\cdot I \\ +4\cdot I \end{array} \left(\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & -1 \\ 0 & \boxed{7} & -\frac{7}{2} & 0 \\ 0 & -6 & 3 & 0 \end{array}\right) :7$$

$$\left(\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & -1 \\ 0 & \boxed{1} & -\frac{1}{2} & 0 \\ 0 & -6 & 3 & 0 \end{array}\right) \begin{array}{l} +2\cdot II \\ \\ +6\cdot II \end{array} \; ! \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & -1 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & \boxed{1} & t \end{array}\right) \begin{array}{l} +\frac{1}{2}\cdot III \\ +\frac{1}{2}\cdot III \\ \\ \end{array}$$

$\rightarrow 0 \cdot x_3 = 0$

$1 \cdot x_3 = t$

$\boxed{t \in \mathbb{R}}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1+\frac{1}{2}t \\ 0 & 1 & 0 & \frac{1}{2}\cdot t \\ 0 & 0 & 1 & t \end{array}\right) \quad \boxed{\begin{array}{l} x_1 = -1+\frac{1}{2}t \\ x_2 = \frac{1}{2}t \\ x_3 = t \end{array}}$$

② Határozzuk meg az alábbi mátrixok LU-felbontását, és determinánsát az LU-felbontás segítségével!

a)

$$A = \begin{pmatrix} 2 & 6 & 3 \\ 1 & 5 & 15 \\ 1 & 1 & -2 \end{pmatrix}$$

$$A_{N\times N} = L_{N\times N} \cdot U_{N\times N}$$

L: normált alsóháromszög mátrix

U: felső-háromszög mátrix

$$\begin{pmatrix} 2 & 6 & 3 \\ 1 & 5 & 15 \\ 1 & 1 & -2 \end{pmatrix} \begin{matrix} \\ -\frac{1}{2}\,\mathrm{I} \\ -\frac{1}{2}\cdot\mathrm{I} \end{matrix}$$

$$\begin{pmatrix} 2 & 6 & 3 \\ 0 & 2 & \frac{27}{2} \\ 0 & -2 & -\frac{7}{2} \end{pmatrix} \begin{matrix} \\ \\ +1\,\mathrm{II} \end{matrix} \qquad -(-1\,\mathrm{II})$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -1 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 2 & 6 & 3 \\ 0 & 2 & \frac{27}{2} \\ 0 & 0 & 10 \end{pmatrix}$$

$$\det A = \det(L \cdot U) = \underbrace{\det L}_{1} \cdot \det U$$

$$= \det U = 2 \cdot 2 \cdot 10 = \underline{\underline{40}}$$

Ell: $A = L \cdot U$

	$L \cdot U$		2	6	3
			0	2	$\frac{27}{2}$
			0	0	10
1	0	0	2	6	3
$\frac{1}{2}$	1	0	1	5	15
$\frac{1}{2}$	-1	1	1	1	-2

✓

Ötlet:

			2	6	3
	$L \cdot U$		0	$u_{2,2}$	$u_{2,3}$
			0	0	$u_{3,3}$
$\boxed{1}$	0	0	2	6	3
$\frac{1}{2}$	1	0	1	5	15
$\frac{1}{2}$	$l_{3,2}$	1	1	1	-2

$\Rightarrow$ $l_{3,2}$

$u_{2,2}, u_{2,3}, u_{3,3}$

d)

$$A = \begin{pmatrix} \boxed{2} & -1 & 2 \\ 4 & -1 & 6 \\ 6 & -7 & 8 \end{pmatrix} \begin{matrix} \\ \boxed{-2} \cdot \mathrm{I} \\ \boxed{-3} \cdot \mathrm{I} \end{matrix}$$

$$\begin{pmatrix} 2 & -1 & 2 \\ 0 & \boxed{1} & 2 \\ 0 & -4 & 2 \end{pmatrix} \begin{matrix} \\ \\ \boxed{+4} \mathrm{II} \end{matrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -4 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 10 \end{pmatrix}$$

$\det A = \det U =$

$= 2 \cdot 1 \cdot 10 = \underline{\underline{20}}$

d)

$$A = \begin{pmatrix} 4 & 4 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -1 & 4 \end{pmatrix} \begin{array}{l} \\ -\frac{1}{2}\cdot \mathrm{I} \\ -\boxed{0}\cdot \mathrm{I} \\ -0\cdot \mathrm{I} \\ -0\cdot \mathrm{I} \end{array} \quad \left(-\frac{2}{4}\cdot \mathrm{I}\right)$$

$$U = \begin{pmatrix} 4 & 4 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & \underline{0} & 2 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{array}{l} \\ \\ -1\,\mathrm{II} \\ -0\cdot \mathrm{III} \\ -0\cdot \mathrm{III} \quad +\frac{1}{4}\,\mathrm{IV} \end{array} \qquad L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & 1 \end{pmatrix}$$

$$\det A = \det = 4\cdot 1\cdot 2\cdot 4\cdot 4 = \underline{\underline{96}}$$

3) Oldjuk meg az alábbi lineáris egyenletrendszert az LU-felbontással!

$$2x_1 - x_2 + 2x_3 = 6$$
$$4x_1 - x_2 + 6x_3 = 20$$
$$6x_1 - 7x_2 + 8x_3 = 16$$

(ea-on megoldottuk, de Gauss-elim.-val!)

<u>Mo:</u>

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 4 & -1 & 6 \\ 6 & -7 & 8 \end{pmatrix}_{3\times 3}, \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{3\times 1}, \quad \underline{b} = \begin{pmatrix} 6 \\ 20 \\ 16 \end{pmatrix}_{3\times 1}$$

$\rightarrow \quad A \cdot \underline{x} = \underline{b}$ $\qquad 3\times\cancel{3} \quad \cancel{3}\times 1$

$(L \cdot U) \cdot \underline{x} = \underline{b}$

$\rightarrow \quad L \cdot (\underbrace{U \cdot \underline{x}}_{\underline{y}}) = \underline{b}$

$\boxed{L \cdot \underline{y} = \underline{b}}$

0. lépés: A LU-felbontása

1. lépés: $L \cdot \underline{y} = \underline{b} \Rightarrow \underline{y}$

2. lépés: $U \cdot \underline{x} = \underline{y} \Rightarrow \underline{x}$

0. lépés: LU-felbontás (ld. 2/d feladat)

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -4 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 10 \end{pmatrix}$$

1. lépés: $L \cdot \underline{y} = \underline{b}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 2 & 1 & 0 & 20 \\ 3 & -4 & 1 & 16 \end{array}\right) \begin{array}{l} \\ -2\,\mathrm{I} \\ -3\,\mathrm{I} \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 8 \\ 0 & -4 & 1 & -2 \end{array}\right) \begin{array}{l} \\ \\ +4\cdot\mathrm{II} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 30 \end{array}\right) \qquad \underline{y} = \begin{pmatrix} 6 \\ 8 \\ 30 \end{pmatrix}$$

2. lépés : $\underline{\underline{U}} \cdot \underline{x} = \underline{y}$

$$\left(\begin{array}{ccc|c} 2 & -1 & 2 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right) \begin{array}{l} -2\cdot\underline{\text{III}} \\ -2\cdot\underline{\text{III}} \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l} +\underline{\text{II}} \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|c} \boxed{\overset{1}{\cancel{2}}} & 0 & 0 & \overset{1}{\cancel{2}} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l} :2 \\ \\ \end{array}$$

$$\underline{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$x_1 = 1$

$x_2 = 2$

$x_3 = 3$

c)

$A =$