1. Gyakorlat

2024. 02. 07.

① Oldjuk meg Gauss(-Jordan) eliminációval az alábbi lineáris egyenletrendszereket!

a) $$\begin{aligned} 2x_1 + x_2 + x_3 &= 4 \\ 3x_1 - 2x_2 - x_3 &= 2 \\ x_1 + 4x_2 + 3x_3 &= 4 \end{aligned}$$

Mo:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & -1 \\ 1 & 4 & 3 \end{pmatrix}_{3\times 3} ; \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{3\times 1} ; \quad \underline{b} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}_{3\times 1}$$

$$A \cdot \underline{x} = \underline{b}$$

Megoldások száma? Mit várunk?

$$\det A = \left| \begin{array}{ccc|cc} 2 & 1 & 1 & 2 & 1 \\ 3 & -2 & -1 & 3 & -2 \\ 1 & 4 & 3 & 1 & 4 \end{array} \right| =$$

$$= \left[2\cdot(-2)\cdot 3 + 1\cdot(-1)\cdot 1 + 1\cdot 3\cdot 4\right] - \left[1\cdot(-2)\cdot 1 + 2\cdot(-1)\cdot 4 + 1\cdot 3\cdot 3\right] = 0 \Rightarrow$$

$\Rightarrow$ megoldások száma nem egy: $\dfrac{0}{\infty}$ sok

$$\left(\begin{array}{ccc|c} \boxed{2} & 1 & 1 & 4 \\ 3 & -2 & -1 & 2 \\ 1 & 4 & 3 & 4 \end{array}\right)$$

$$\begin{array}{l} \left(\begin{array}{ccc|c} 1 & 4 & 3 & 4 \\ 3 & -2 & -1 & 2 \\ 2 & 1 & 1 & 4 \end{array}\right) \end{array} \begin{array}{l} \\ -3\cdot \mathrm{I} \\ -2\cdot \mathrm{I} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 4 & 3 & 4 \\ 0 & \boxed{-14} & -10 & -10 \\ 0 & -7 & -5 & -4 \end{array}\right) \begin{array}{l} \\ \\ -\frac{1}{2}\,\mathrm{II} \end{array}$$

$\Rightarrow$ nincs mo.

$$\left(\begin{array}{ccc|c} 1 & 4 & 3 & 4 \\ 0 & -14 & -10 & -10 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$\Rightarrow$ nincs mo. !!!

b) $x_1 - x_2 + 3x_3 = 1$

$2x_1 + x_2 - 2x_3 = 3$

$x_1 + 5x_2 - 13x_3 = 2$

<u>Mo:</u>

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & -2 \\ 1 & 5 & -13 \end{pmatrix}_{3\times 3} ; \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{3\times 1} ; \quad \underline{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}_{3\times 3}$$

$$A \cdot \underline{x} = \underline{b}$$

Hány megoldást várhatunk?

$$\det A = \left| \begin{array}{ccc|cc} 1 & -1 & 3 & 1 & -1 \\ 2 & 1 & -2 & 2 & 1 \\ 1 & 5 & -13 & 1 & 5 \end{array} \right| =$$

$$= \left[1 \cdot 1 \cdot (-13) + (-1)(-2) \cdot 1 + 3 \cdot 2 \cdot 5 \right] -$$

$$- \left[3 \cdot 1 \cdot 1 + 1 \cdot (-2) \cdot 5 + (-1) \cdot 2 \cdot (-13) \right] = \underline{\underline{0}}$$

$\Longrightarrow$ 0 mo.

∞ sok mo.

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 2 & 1 & -2 & 3 \\ 1 & 5 & -13 & 2 \end{array}\right) \begin{array}{l} \\ -2\cdot\mathrm{I} \\ -\mathrm{I} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & \boxed{3} & -8 & 1 \\ 0 & 6 & -16 & 1 \end{array}\right) \begin{array}{l} \\ \Longrightarrow \text{ nüis mo.} \\ -2\cdot\mathrm{II} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 3 & -8 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \Longrightarrow \text{ nües mo.}$$

c) $x_1 + x_2 - x_3 = 1$

$3x_1 - x_2 - x_3 = -1$

$5x_1 + x_1 - 3x_3 = 1$

<u>Mo:</u>

$$\begin{array}{c}\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & -1 & -1 & -1 \\ 5 & 1 & -3 & 1 \end{array}\right)\end{array}\begin{array}{l} \\ -3\cdot \mathrm{I} \\ -5\cdot \mathrm{I}\end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & \boxed{-4} & 2 & -4 \\ 0 & -4 & 2 & -4 \end{array}\right)\begin{array}{l} \\ :-4 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & -4 & 2 & -4 \end{array}\right)\begin{array}{l} -\mathrm{II} \\ \\ +4\cdot \mathrm{II}\end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 1 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} \end{array}\right)$$

t

$0 \cdot x_3 = 0$

$\boxed{x_3 = t}\,,\ t \in \mathbb{R}$

$$
\left(\begin{array}{ccc|c}
1 & 0 & -\frac{1}{2} & 0 \\
0 & 1 & -\frac{1}{2} & 1 \\
0 & 0 & 1 & t
\end{array}\right)
\begin{array}{l}
+\frac{1}{2}\,\underline{\mathrm{III}} \\
+\frac{1}{2}\,\underline{\mathrm{III}} \\

\end{array}
$$

$$
\left(\begin{array}{ccc|c}
1 & 0 & 0 & \frac{1}{2}t \\
0 & 1 & 0 & 1+\frac{1}{2}t \\
0 & 0 & 1 & t
\end{array}\right)
$$

$$
\underline{x} = \begin{pmatrix} \frac{1}{2}t \\ 1+\frac{1}{2}t \\ t \end{pmatrix} ; \; t \in \mathbb{R}
$$

d) $2x_1 - x_2 - x_3 = -2$

$x_1 + x_2 - 2x_3 = -7$

$x_1 - 3x_2 + 2x_3 = 9$

$$\left(\begin{array}{ccc|c} \boxed{2} & -1 & -1 & -2 \\ 1 & 1 & -2 & -7 \\ 1 & -3 & 2 & 9 \end{array}\right)$$

$$\begin{array}{cc} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -7 \\ 2 & -1 & -1 & -2 \\ 1 & -3 & 2 & 9 \end{array}\right) & \begin{array}{l} \\ -2\mathrm{I} \\ -\mathrm{I} \end{array} \end{array}$$

$$\begin{array}{cc} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -7 \\ 0 & \boxed{-3} & 3 & 12 \\ 0 & -4 & 4 & 16 \end{array}\right) & \begin{array}{l} \\ :(-3) \\ \\ \end{array} \end{array}$$

$\Rightarrow$ végtelen sok mo.

$$\begin{array}{cc} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -7 \\ 0 & 1 & -1 & -4 \\ 0 & -4 & 4 & 16 \end{array}\right) & \begin{array}{l} -\mathrm{II} \\ \\ +4\cdot\mathrm{II} \end{array} \end{array}$$

$$
\begin{array}{c}
\left(\begin{array}{ccc|c}
1 & 0 & -1 & -3 \\
0 & 1 & -1 & -4 \\
\boxed{0 \quad 0 \quad 0} & & & \boxed{0}
\end{array}\right) \\
\begin{array}{cccc}
 & & 1 & t
\end{array}
\end{array}
$$

$$0 \cdot x_3 = 0$$

$$1 \cdot x_3 = t \; ; \; t \in \mathbb{R}$$

$$
\left(\begin{array}{ccc|c}
1 & 0 & -1 & -3 \\
0 & 1 & -1 & -4 \\
0 & 0 & 1 & t
\end{array}\right)
\begin{array}{l}
+\,\mathrm{III} \\
+\,\mathrm{III} \\

\end{array}
$$

$$
\left(\begin{array}{ccc|c}
1 & 0 & 0 & t-3 \\
0 & 1 & 0 & t-4 \\
0 & 0 & 1 & t
\end{array}\right)
$$

$$
\underline{X} = \begin{pmatrix} t-3 \\ t-4 \\ t \end{pmatrix}, \; t \in \mathbb{R}
$$

2.) Határozzuk meg az alábbi mátrixok LU-felbontását (ha létezik). Határozzuk meg a mátrixok determinánsát az LU-felbontás segítségével!

a)
$$A = \begin{pmatrix} \boxed{2} & 4 & -2 \\ 3 & 1 & 1 \\ 4 & 1 & 2 \end{pmatrix} \begin{matrix} \\ \underbrace{-\tfrac{3}{2}}\,\mathrm{I} \\ \underbrace{-2}\cdot\mathrm{I} \end{matrix}$$

$\left[\, A = L \cdot U \,\right]$

normált alsóháromszög mátrix

felsőháromszög mátrix

Mo: (Gauss-el.)

[tilos: • sorcsere
• sort konstanssal osztani, szorozni]

$$\begin{pmatrix} 2 & 4 & -2 \\ 0 & \boxed{-5} & 4 \\ 0 & -7 & 6 \end{pmatrix} \underbrace{-\tfrac{7}{5}}\,\mathrm{II}$$

$$- \frac{-7}{-5} \cdot \mathrm{II}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 2 & \frac{7}{5} & 1 \end{pmatrix}_{3\times3}$$

$$U = \begin{pmatrix} 2 & 4 & -2 \\ 0 & -5 & 4 \\ 0 & 0 & 0{,}4 \end{pmatrix}$$

hf: ell: $L \cdot U = A$

$$\det A = \det(L \cdot U) = \underbrace{\det L}_{1} \cdot \det U =$$

$$= \det U = \underbrace{2 \cdot (-5) \cdot 0{,}4} = \underline{\underline{-4}}$$

b)

$$B = \begin{pmatrix} 4 & 4 & -2 \\ 2 & -1 & 1 \\ -2 & 1 & 2 \end{pmatrix}_{3\times 3}$$

$B = L \cdot U$

L: normált alsó háromszög mátrix

U: felső háromszög mátrix

Mo: (LU-felb. Gauss-el. : tilos:

- sorcsere
- sor szorzása konstanssal

$$\begin{pmatrix} \boxed{4} & 4 & -2 \\ 2 & -1 & 1 \\ -2 & 1 & 2 \end{pmatrix} \begin{matrix} -\frac{1}{2} \cdot I \quad \left[-\frac{2}{4} \cdot I\right] \\ +\frac{1}{2} \cdot I \quad \left[-\frac{-2}{4} \cdot I\right] \end{matrix}$$

$$\begin{pmatrix} 4 & 4 & -2 \\ 0 & \boxed{-3} & 2 \\ 0 & 3 & 1 \end{pmatrix} + 1 \cdot II$$

$$\rightarrow L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -1 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & 4 & -2 \\ 0 & -3 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

Hf: $L \cdot U = B$ ellenőrzése

$$\boxed{\det B =} \det(L \cdot U) = \underbrace{\det L}_{1 \cdot 1 \cdot 1} \cdot \det U =$$

$$\boxed{= \det U} = 4 \cdot (-3) \cdot 3 = \underline{\underline{-36}}$$

c)

$$C = \begin{pmatrix} \boxed{0{,}5} & 1 & -0{,}5 \\ 1 & -2 & 1 \\ -0{,}5 & 1 & 2 \end{pmatrix} \begin{matrix} -2 \cdot \mathrm{I} \\ +1 \cdot \mathrm{I} \end{matrix}$$

$$\begin{pmatrix} 0{,}5 & 1 & -0{,}5 \\ 0 & \boxed{-4} & 2 \\ 0 & 2 & 1{,}5 \end{pmatrix} \begin{matrix} \\ \\ -\frac{2}{-4} \mathrm{II} \end{matrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{1}{2} & 1 \end{pmatrix}$$

$$+\frac{1}{2} \cdot \mathrm{II}$$

$$U = \begin{pmatrix} 0{,}5 & 1 & -0{,}5 \\ 0 & -4 & 2 \\ 0 & 0 & 2{,}5 \end{pmatrix}$$

Ell:

$$\begin{array}{ccc|ccc} & & & \frac{1}{2} & 1 & -\frac{1}{2} \\ & L \cdot U & & 0 & -4 & 2 \\ & & & 0 & 0 & \frac{5}{2} \\ \hline 1 & 0 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} \\ 2 & 1 & 0 & 1 & -2 & +1 \\ -1 & -\frac{1}{2} & 1 & -\frac{1}{2} & 1 & 2 \end{array}$$

$$\det C = \det U = 0{,}5 \cdot (-4) \cdot 2{,}5 = \underline{\underline{-5}}$$

d)

$$D = \begin{pmatrix} \boxed{3} & 1 & 2 \\ 1 & -\frac{2}{3} & \frac{5}{3} \\ 2 & \frac{2}{3} & \frac{4}{3} \end{pmatrix} \begin{matrix} \\ \boxed{-\frac{1}{3}}\,\mathrm{I} \\ \boxed{-\frac{2}{3}}\,\mathrm{I} \quad \left[-\frac{2}{3}\,\mathrm{I}\right] \end{matrix}$$

$$U = \begin{pmatrix} 3 & 1 & 2 \\ 0 & \boxed{-1} & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ +\boxed{0}\cdot\mathrm{II} \end{matrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & 0 & 1 \end{pmatrix}$$

$$\det D = \det U = 3 \cdot (-1) \cdot 0 = 0$$

3. Oldjuk meg az alábbi lineáris egyenletrendszereket az LU-felbontás segítségével!

a)
$$\begin{aligned} 2x_1 - x_2 + 2x_3 &= 6 \\ 4x_1 - x_2 + 6x_3 &= 20 \\ 6x_1 - 7x_2 + 8x_3 &= 16 \end{aligned}$$

(ld. előadás feladata)

b)
$$\begin{aligned} 2x_1 + 6x_2 + 3x_3 &= 1 \\ x_1 + 5x_2 + 15x_3 &= 14 \\ x_1 + x_2 - 2x_3 &= -3 \end{aligned}$$

hf.

eredm.: $\underline{x} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Emlékeztető:

→ $A \cdot \underline{x} = \underline{b}$

0. lépés: A mátrix LU-felbontása

$$(L \cdot U) \cdot \underline{x} = \underline{b}$$

$$L \cdot \underbrace{(U \cdot \underline{x})}_{\underline{y}} = \underline{b}$$

jelölés: $U \cdot \underline{x} = \underline{y}$

1. lépés: $L \cdot \underline{y} = \underline{b} \implies \underline{y}$

2. lépés: $U \cdot \underline{x} = \underline{y} \implies \underline{x}$

a) $A = \begin{pmatrix} 2 & -1 & 2 \\ 4 & -1 & 6 \\ 6 & -7 & 8 \end{pmatrix}$, $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 6 \\ 20 \\ 16 \end{pmatrix}$

0. lépés A mátrix LU-felbontása

ld. előadás

$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -4 & 1 \end{pmatrix}$; $U = \begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 10 \end{pmatrix}$

1. lépés $L \cdot \underline{y} = \underline{b} \implies \underline{y}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 2 & 1 & 0 & 20 \\ 3 & -4 & 1 & 16 \end{array}\right) \begin{array}{l} \\ -2\,\mathrm{I} \\ -3\cdot\mathrm{I} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 8 \\ 0 & -4 & 1 & -2 \end{array}\right) \begin{array}{l} \\ \\ +4\cdot\underline{\mathrm{II}} \end{array} \implies \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 30 \end{array}\right)$$

$$\Uparrow$$

$$\underline{y} = \begin{pmatrix} 6 \\ 8 \\ 30 \end{pmatrix}$$

2. lépés: $\quad U\underline{x} = \underline{y} \implies \underline{x}$

$$\left(\begin{array}{ccc|c} 2 & -1 & 2 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & \boxed{1\cancel{0}} & 3\cancel{0} \end{array}\right) \begin{array}{l} -2\cdot \mathrm{III} \\ -2\cdot \mathrm{III} \\ :10 \end{array}$$

$$\begin{array}{cccc} & & 1 & 3 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l} +\mathrm{II} \\ \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & & & 1 \end{array}$$

$$\left(\begin{array}{ccc|c} \boxed{\cancel{2}} & 0 & 0 & \cancel{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l} :2 \\ \\ \\ \end{array}$$

$$\underline{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}; \qquad \begin{array}{l} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{array}$$